PSO 2

Question 1

(Recursion Tree) Give a big-O closed form for each of the following recurrences. (Assume that $T(x) = 1$ for any $x \leq 1$.)

(1) $T(n) = 2T(n/4) + \sqrt{n}$ $= 2(2T(n/16) + \sqrt{(n/4)}) + \sqrt{n} = 2(2(2T(n/64) + \sqrt{n/16}) + \sqrt{n/4}) + \sqrt{n}$

$\sqrt{n}$

$2 = \sqrt{4}$

(2) $T(n) = T(n/2) + T(n/3) + T(n/6) + n$

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Warning: Solving this $T(n)$ using iterations is a bad idea!

… kind of, we will see that trees help us organize better!

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2. Find the cost at the ith level and the number of levels
3. Derive the sum and closed form

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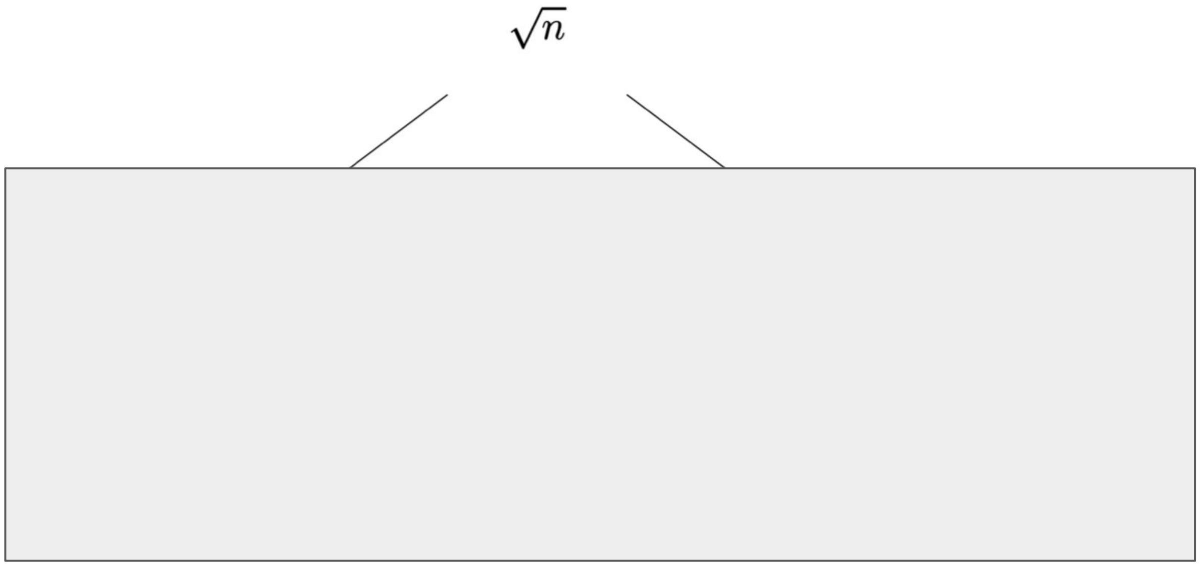

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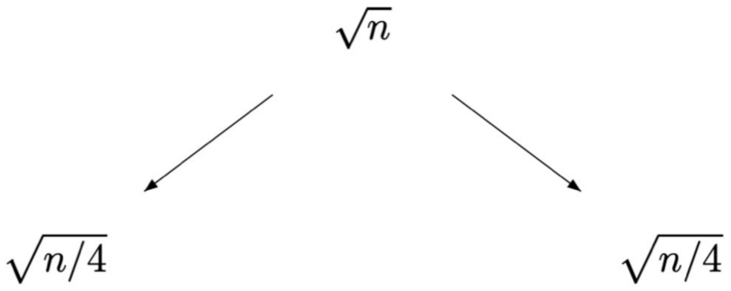

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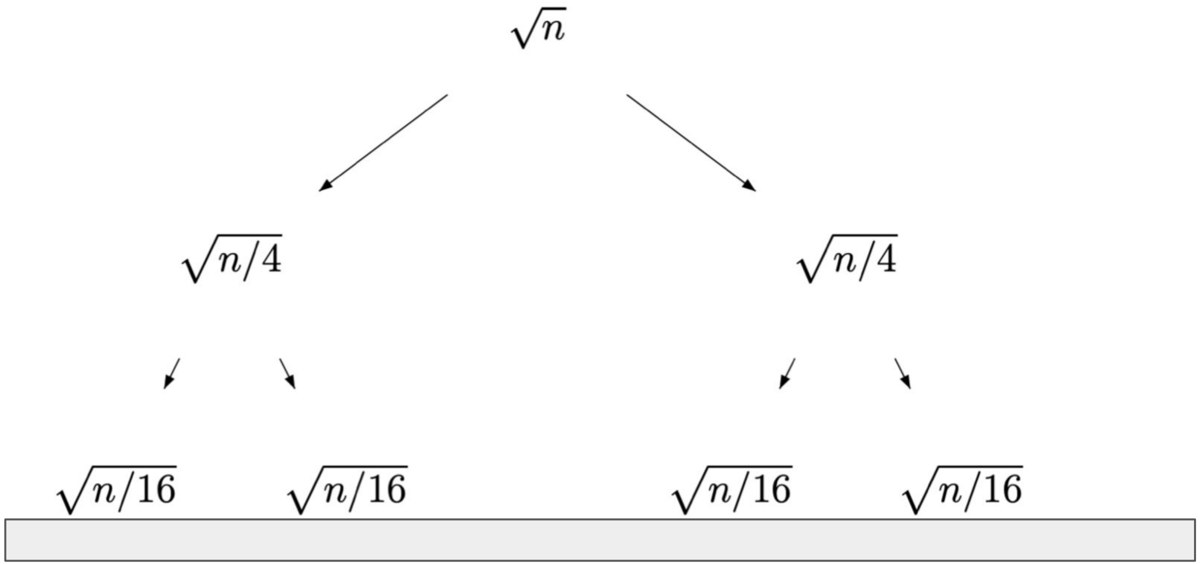

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$T(n) = T(n/c) \to \log_c n$ levels.

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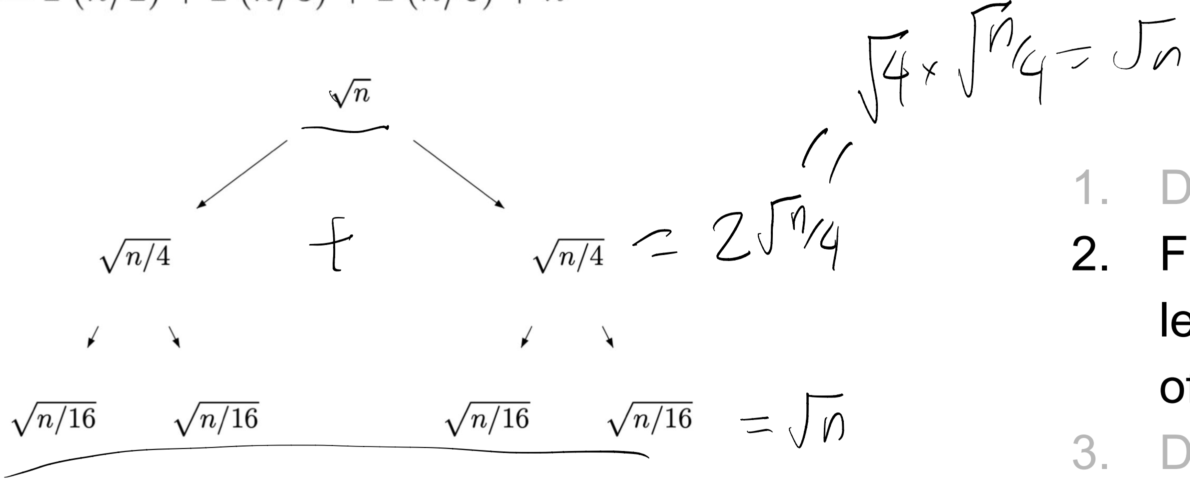

1. Draw out the tree
2. **Find the cost at the ith level and the number of levels**
3. Derive the sum and closed form

Cost at first level: $\sqrt{n}$

Cost at second level: $\sqrt{n/4} + \sqrt{n/4} = 2\sqrt{n/4} = \sqrt{n}$

Cost at ith level: $2^i \times \sqrt{n/4^i} = \sqrt{4^i}\sqrt{n/4^i} = \sqrt{n}$

levels: $\log_4 n$

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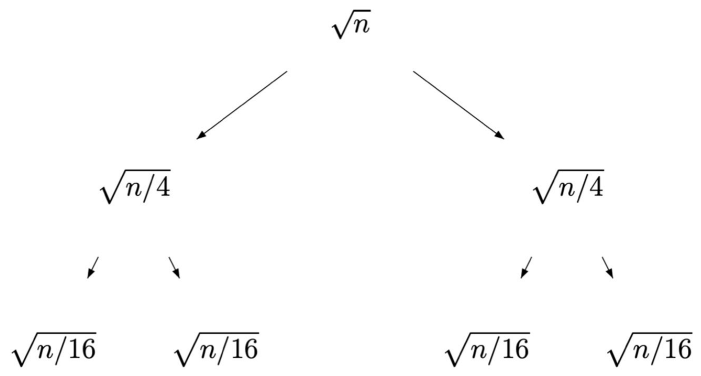

Cost at ith level: $\sqrt{n}$

Number of levels: $\log_4 n$

$$\sqrt{n} \times \log_4 n = \sum_{i=\cancel{1}\,0}^{\#\text{levels}} \text{work @ ith level}$$

↑ ith level

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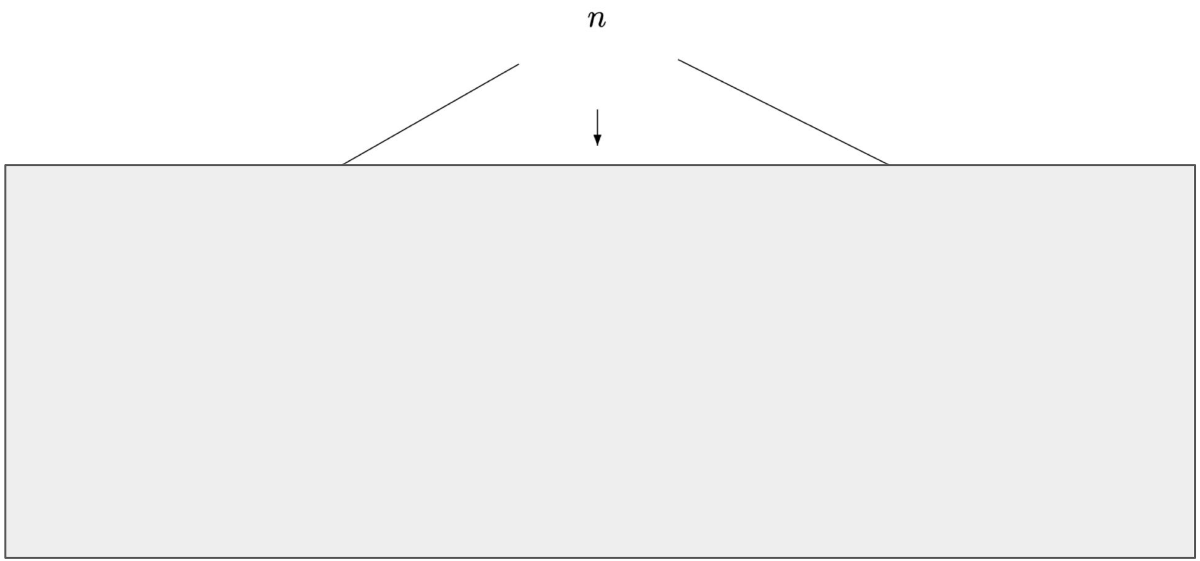

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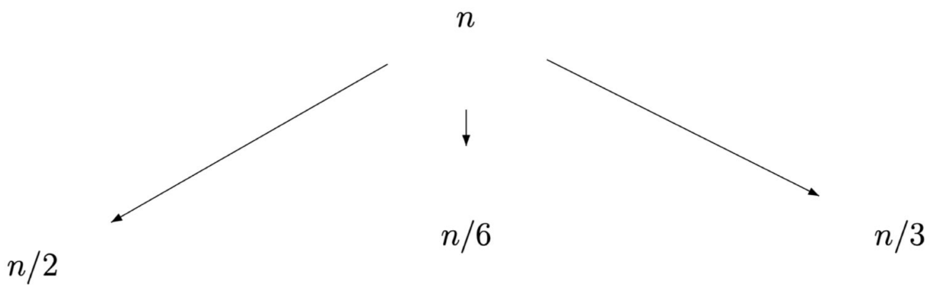

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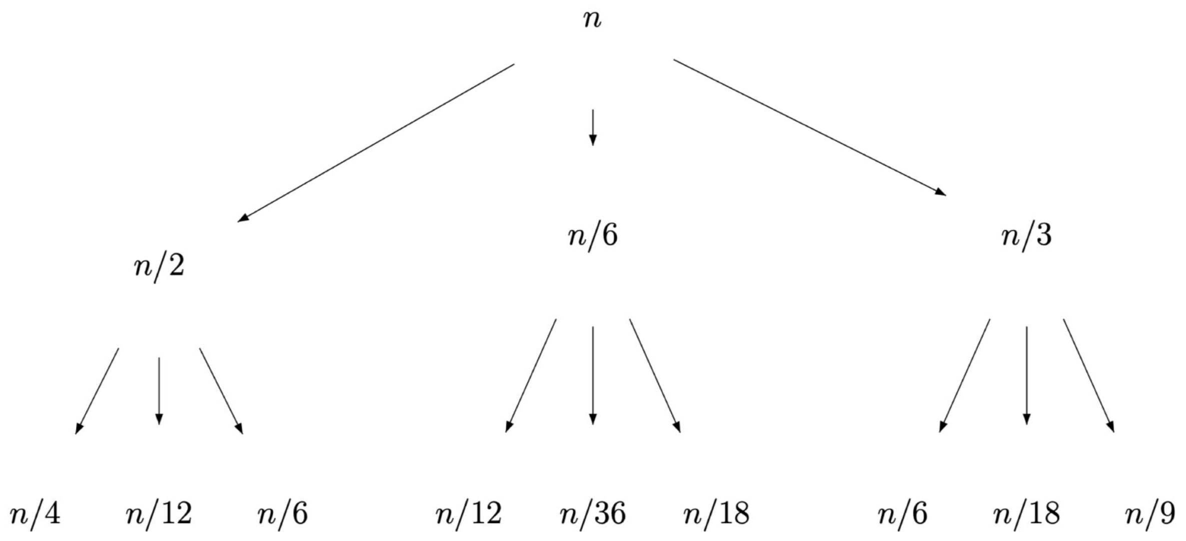

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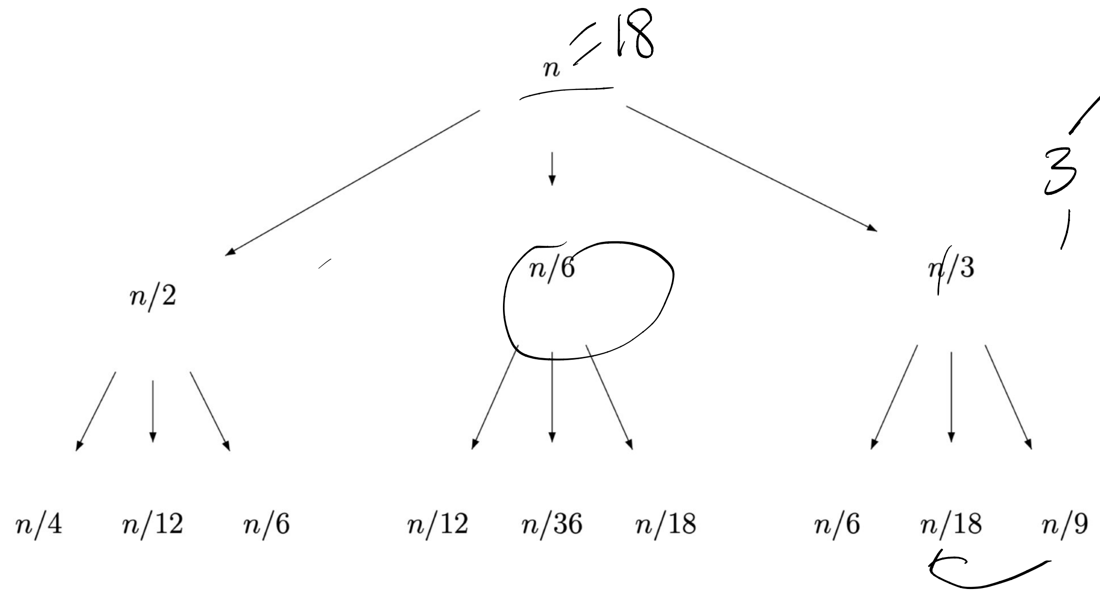

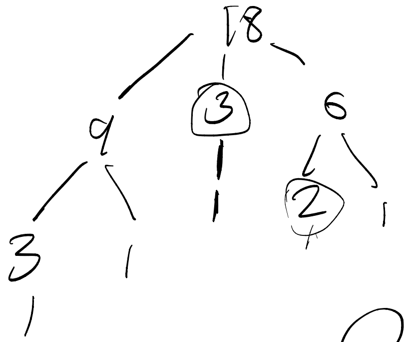

$n\log_6 n - n\log_2 n$

$\in O(n\log_2 n)$

$O(n\log_2 n)$

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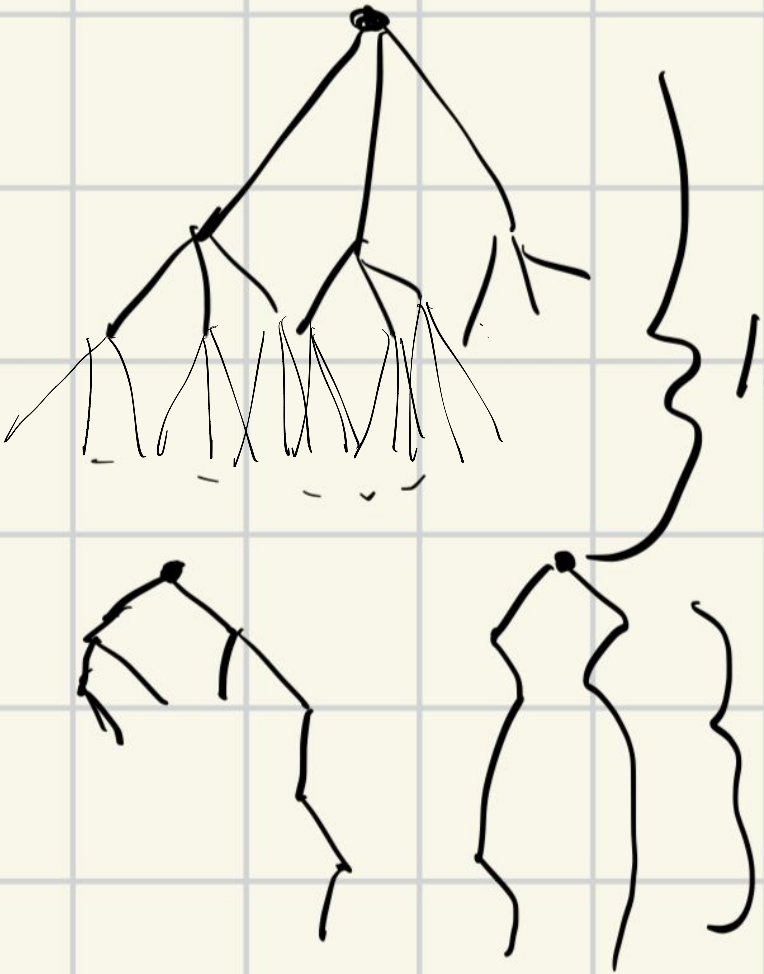
Cost at first level: n
Cost at second level: $n/2 + n/6 + n/3 = n$
Cost at ith level: n

\# levels: $\log_2 n$



$\log_6 n$

$\lg n - \lg_6 n \leq \lg n$

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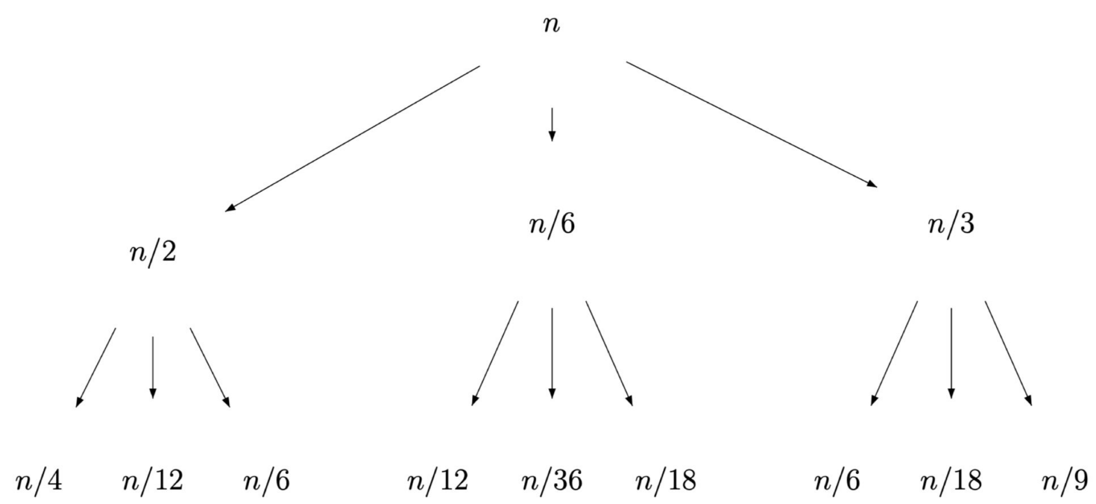

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Cost at ith level: n

Number of levels: $O(\lg n)$

Question 2

(Change a Variable) Give a big-O closed form for the following recurrence.

$$T(n) = 2T(\sqrt{n}) + \log n$$

$n = 2$ $T(n) = 2T(1.41\ldots) + \log 1.4\ldots$

$+ \log 2$

$= 2T(1.1\ldots\ldots)$

$\vdots$

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What is the problem with a tree?

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$T(n^c)$

We usually like recurrences of this form

$$S(n) = \alpha S(n/\beta) + f(n),$$

E.g question 1 recurrences

Solution: Variable change! But to what value? (hint: $\sqrt{n} = n^{1/2}$)

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Change variable: m = $\log n$

$$T(2^m) = 2T(2^{m/2}) + m$$

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Change variable: m = log n

$$T(2^m) = 2T(2^{m/2}) + m.$$

Change equation: S(m) = $T(2^m)$ $\rightarrow$ $S(m) = 2S(m/2) + m.$

$O(m \log m)$

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Change equation: $S(m) = T(2^m)$

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This is just merge sort! O(mlogm) = O(log n * (log log n))

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(Algorithm Design) Describe a $\Theta(n \log n)$ algorithm that, given a set S of n integers and another integer x, determines whether or not there exist two elements in S whose sum is exactly x.

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Naive n^2 strategy?

S $\qquad$ x

$[1, 5, 2, 3, 4]$ $\qquad$ 5

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S	x
$[1, 5, 2, 3, 4]$	5

$$\binom{n}{2} = \frac{n(n-1)}{2} \in \Theta(n^2)$$

$\underline{(1,5)}$, $(1,2)$, $(1,3)$, $(1,4)$

$(5,2)$, $(5,3)$, $(5,4)$

$(2,3)$, $(2,4)$

$(3,4)$

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What can we do in nlogn time?

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What can we do in nlogn time? Sort

Ok now that elements are sorted,

What do we do?

S $\quad$ x

$[1, 5, 2, 3, 4]$ $\quad$ 5

$[1, 2, 3, 4, 5]$ $\quad$ 5

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Idea: find pairs smarter

S $\qquad x$

$[1, 5, 2, 3, 4] \qquad 5$

$[1, 2, 3, 4, 5] \qquad 5$

$O(n)$

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<u>Idea</u>: find pairs smarter

Hold a left,right pointer, calculate sum

If sum > x: move right pointer

If sum < x: move left pointer

S $\qquad$ x

$[1, 5, 2, 3, 4]$ $\qquad$ 5

$[1, 2, 3, 4, 5]$ $\qquad$ 5

1 1 3 7

$\Theta(n \log n) + \Theta(n) = \Theta(n \log n)$

Alg(A):

call 2 Alg(A[2:]) + $\sum_{i=0}^{n-1} A[i]$

$T(n) = 2T(n-1) + n$

$T(1) = 1$

n

$n-1 \quad n-1 = 2n-2$

$n-2 \quad n-2 \quad n-2 \quad n-2 = 4n-8$

n

$n-1$

$n-2$

$\vdots$

1

$$\sum_{i=1}^{n-1} n-i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} 2^i(n-2) =$$

n

$n-1 \qquad n-1 = 2n-2$ $\qquad 1$

$n-2 \qquad n-2 \qquad n-2 \qquad n-2 = 4n-8$ $\qquad 1$

$$\sum_{i=1}^{n} 2^i (n-i) = n \underline{\sum_{i=1}^{n} 2^i} - \sum i \, 2^i$$

$$= n\left(2^{n+1}-1\right) - 2n$$

$$\in O(n 2^n)$$

Question 4

let $n=N$

(Linked List) Consider a sorted circular doubly linked list of N numbers where the head element points to the smallest element in the list. Provide the asymptotic complexity in big-Θ with a brief explanation (including assumptions and analysis for each case, if there is more than one) for the following operations.

1. Inserting an element in its sorted position. $\Theta(n)$
2. Finding the smallest element in the list. $\Theta(1)$
3. Finding the 3^{rd} - largest element in the list. If all N elts are unique $\Theta(1)$ — If duplicates $\Theta(n)$
4. Finding the median in the list. $\Theta(n)$

1 2 2 3 4

1 2 3

1 1 1 2 2 3 3 4 4 5

c) Prove $n(2+\sin n\pi/2)$ is $\Theta(n)$
$f(n)$ "

wts: $f(n) \in O(n)$ and $f(n) \in \Omega(n)$

wts: $\exists c>0, n_0 \in \mathbb{N} : cf(n) \leq n$

$cn(2+\sin^{n\pi}/2) = 2cn + c\sin^{n\pi}/2 \leq n$

we see that $\sin(n\pi/2) \in [-1, 1]$

$\Leftarrow \quad 3cn \leq n$

choose $n_0 = 1$
$c = 1/4$

$3/4\, n \leq n$ ✓

$cn \geq n$

$c = 1$