PSO 12

Minimum Spanning Trees, Prim's vs. Kruskal's, Topos == DAG

Slides @ justin-zhang.com/teaching/CS251

Question 1

(Minimum spanning trees)

1. An edge is called a **light-edge** crossing a cut $\mathcal{C} := (S, V - S)$, if its weight is the minimum of any edge crossing the cut. Show that:

- if an edge (u, v) is contained in some MST, then it is a light-edge crossing some cut of the graph.
- the converse is not true, and give a simple counter-example of a connected graph such that there exists a cut $\mathcal{C} := (S, V - S)$, in which (u, v) is a light-edge crossing the cut $\mathcal{C}$ but does not form a MST of the graph.

2. Show that a graph has a unique MST, if for every cut of the graph, there is a unique light-edge crossing the cut. Show that the converse is not true by giving a counter-example.

3. Let T be an MST of a graph $G = (V, E)$, and let $V^{'}$ be a subset of V. Let $T^{'}$ be the subgraph of T induced by $V^{'}$, and let $G^{'}$ be the subgraph of G induced by $V^{'}$. Show that if $T^{'}$ is connected, then $T^{'}$ is an MST of $G^{'}$.

Question 2

(Prim's & Kruskal's algorithm)

1. Suppose that we represent the graph $G = (V, E)$ as an adjacency-matrix. Give a simple implementation of Prim's algorithm for this case that runs in $O(|V|^2)$ time.

2. Suppose that all edge weights in a graph are integers in the range from 1 to $|V|$. How fast can you make Kruskal's algorithm run?

Question 3

(Topological Ordering)

1. Draw a directed acyclic graph $G = (V, E)$ with $|V| = 5$ nodes that has exactly two topological orderings.

2. Prove that G has a topological ordering if and only if G is a DAG.

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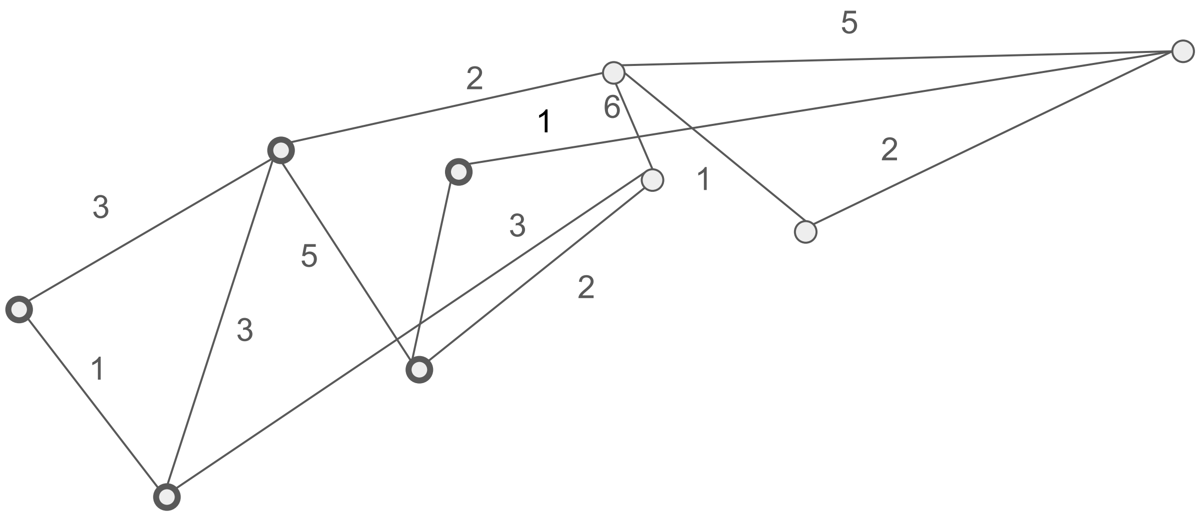

Say I define **C** as

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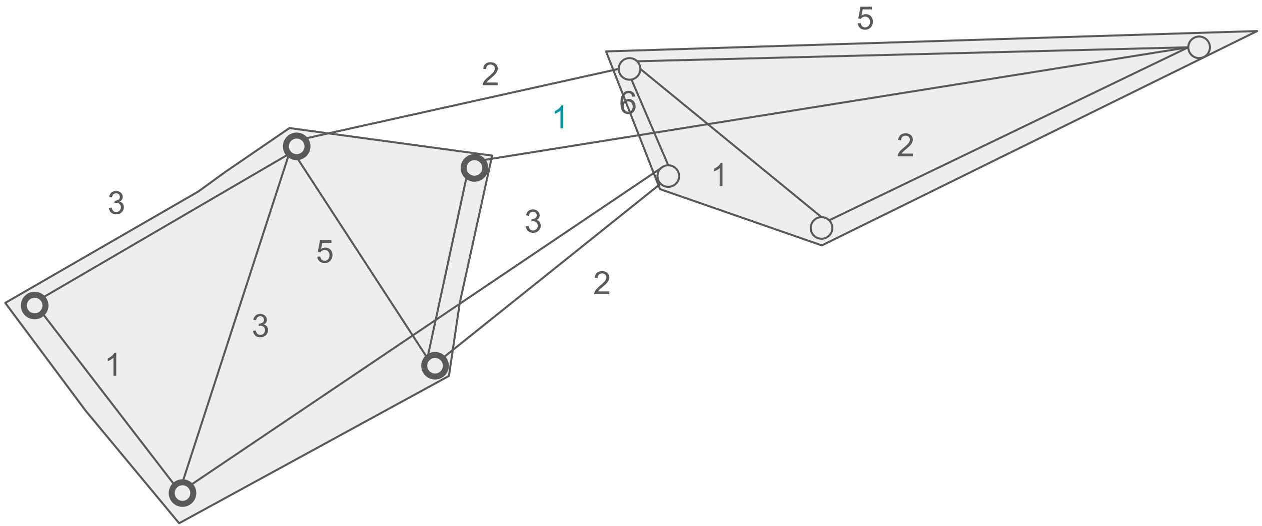

This forms a 'cut'

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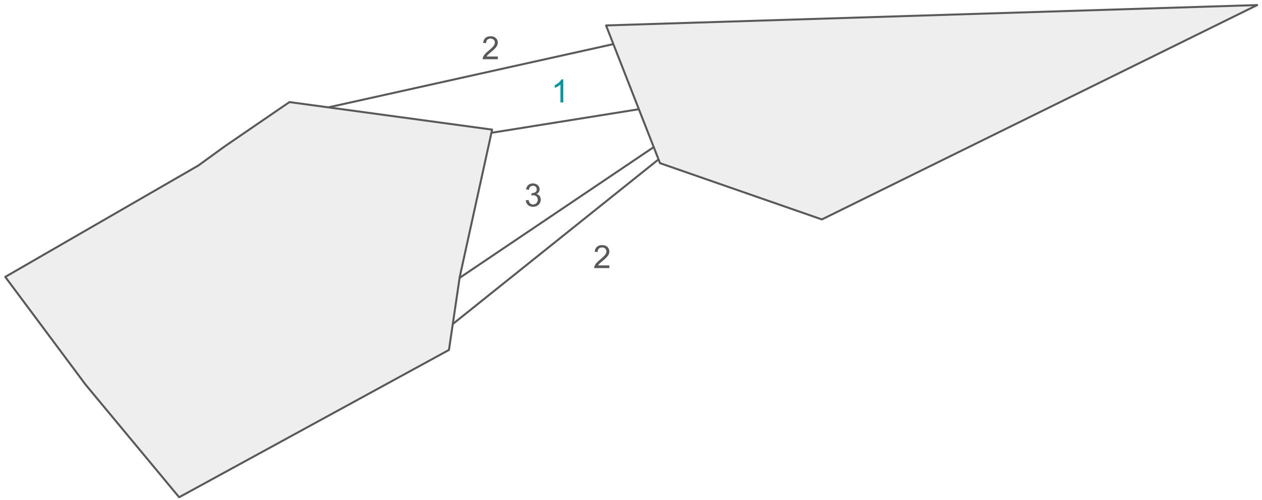
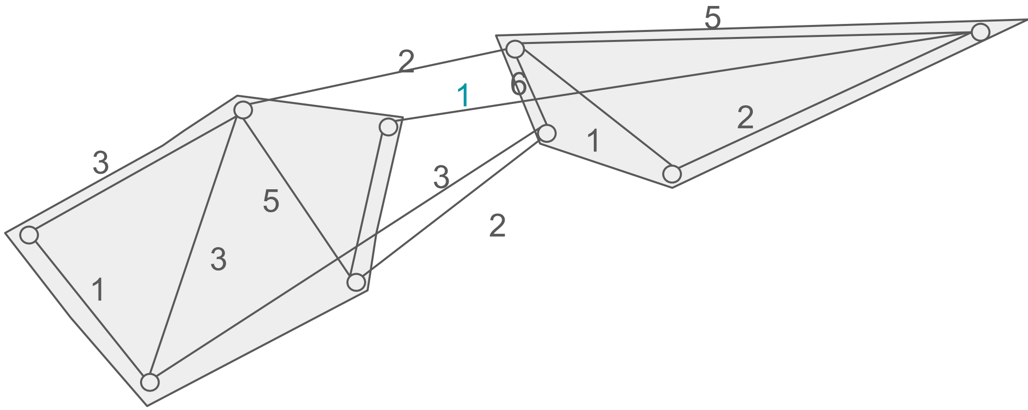


The light edge of this cut has weight 1

- if an edge (u, v) is contained in some MST, then it is a light-edge crossing some cut of the graph.

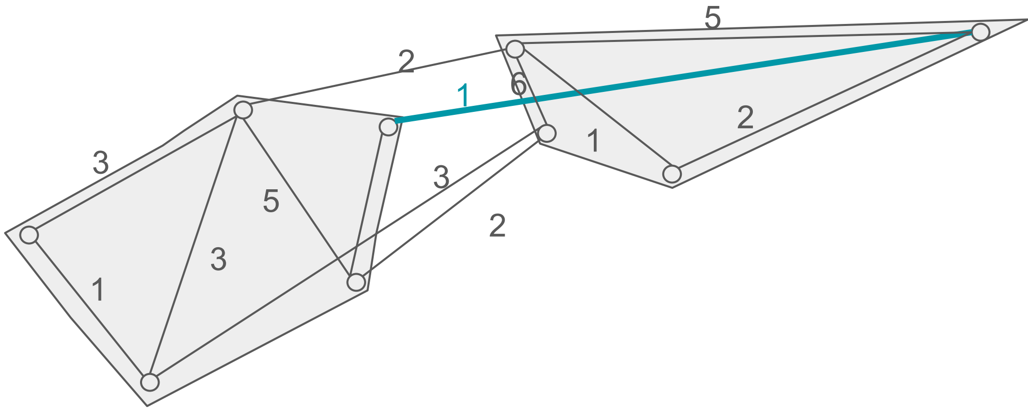
Pf:



- if an edge (u, v) is contained in some MST, then it is a light-edge crossing some cut of the graph.

Pf: AFtSoC e is ~~not in a MST~~ not a light edge

[What happens in the picture?]



- if an edge (u, v) is contained in some MST, then it is a light-edge crossing some cut of the graph.

Suppose e is in the mst

Pf: AFtSoC e is ~~not in a MST~~ not a light.

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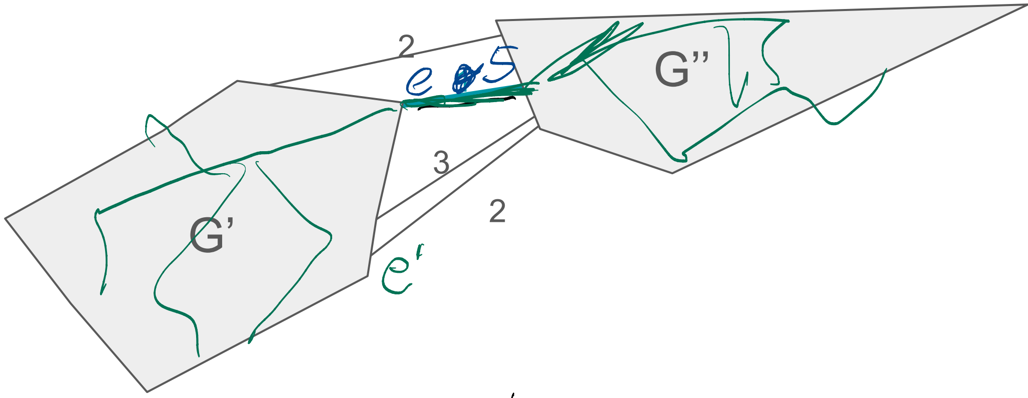


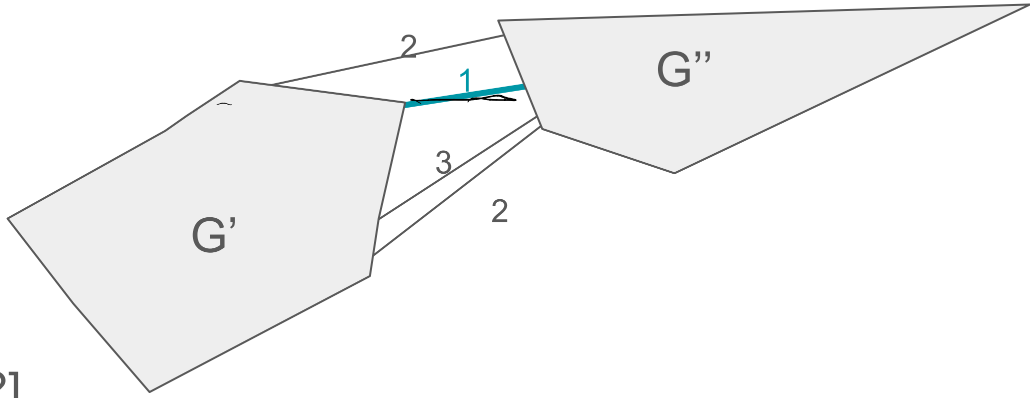
I can get a lighter mst by instead taking edge e' ?

• if an edge (u, v) is contained in some MST, then it is a light-edge crossing some cut of the graph.

Pf: AFtSoC e ~~is not in a MST~~ is not a light edge

In an MST, G' and G'' must be connected.

[How can we get our contradiction?]



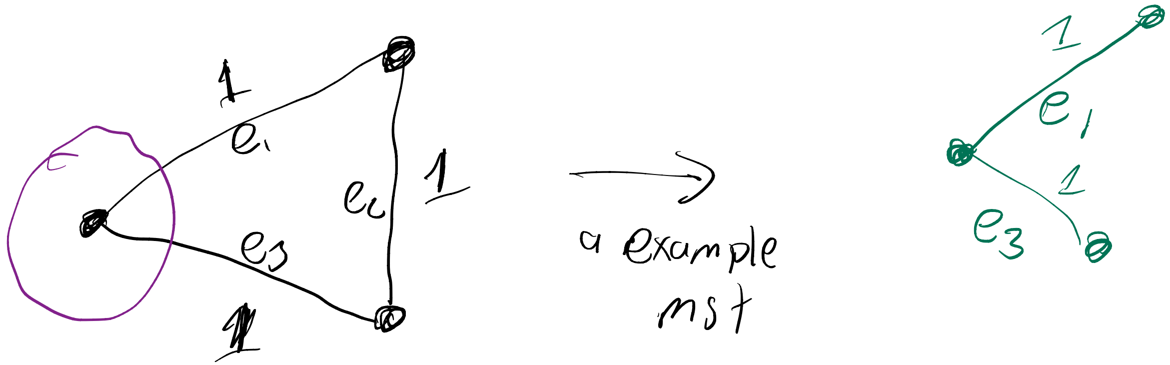
Question 1

(Minimum spanning trees)

1. An edge is called a **light-edge** crossing a cut $\mathcal{C} := (S, V - S)$, if its weight is the minimum of any edge crossing the cut. Show that:

"If e is the light edge of some cut, then it is in *every* MST."

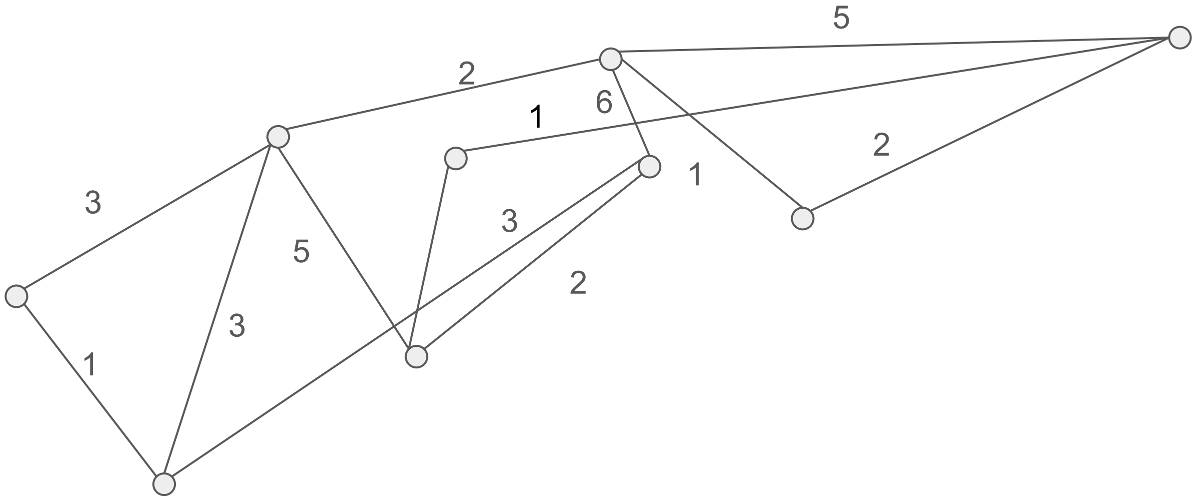
Show that this is false.



2. Show that a graph has a unique MST, if for every cut of the graph, there is a unique light-edge crossing the cut. Show that the converse is not true by giving a counter-example.

Suppose each cut has a unique light edge. **WTS**: the graph has a unique MST

Proof by picture!



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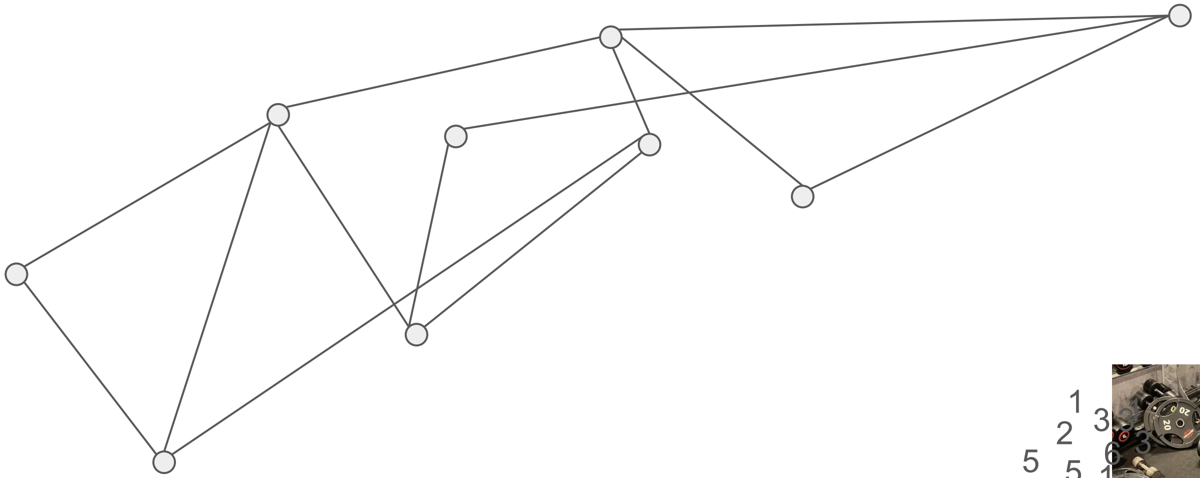

(Me and my bois have taken all the weights off the graph (we need them for our super set))

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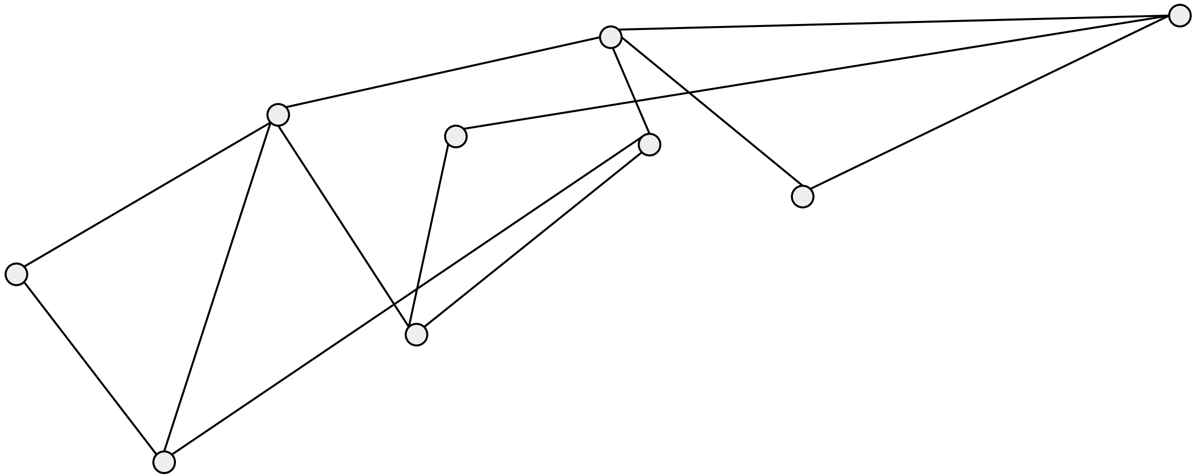

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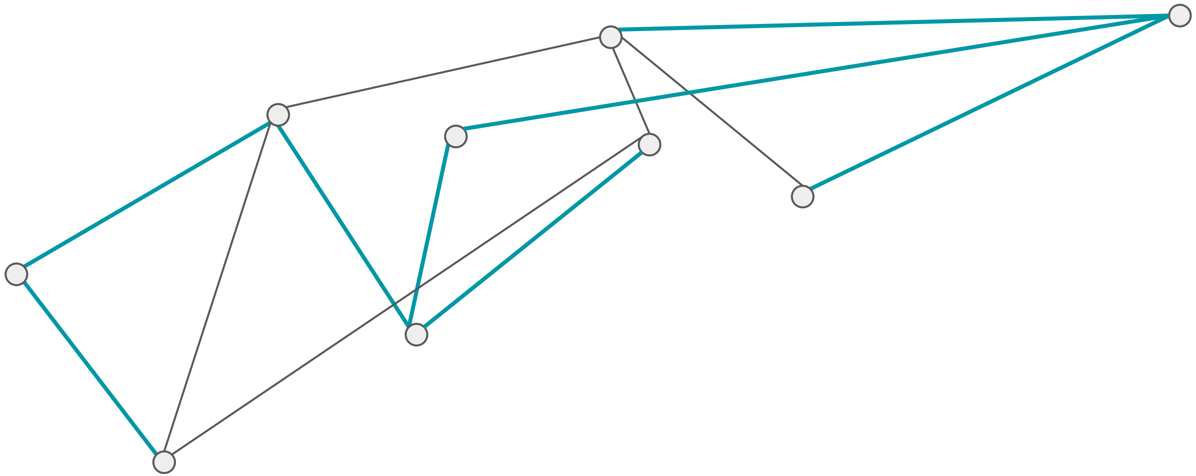

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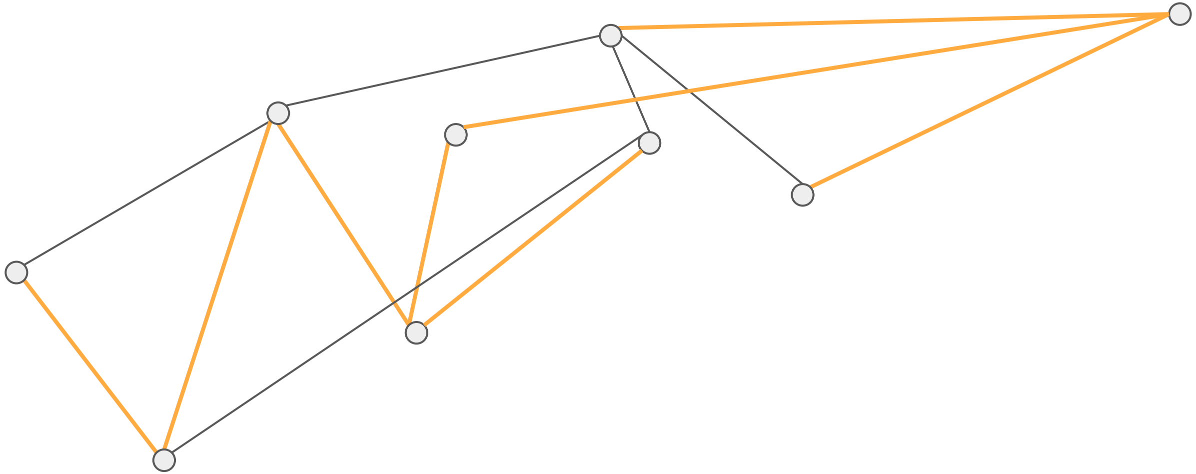

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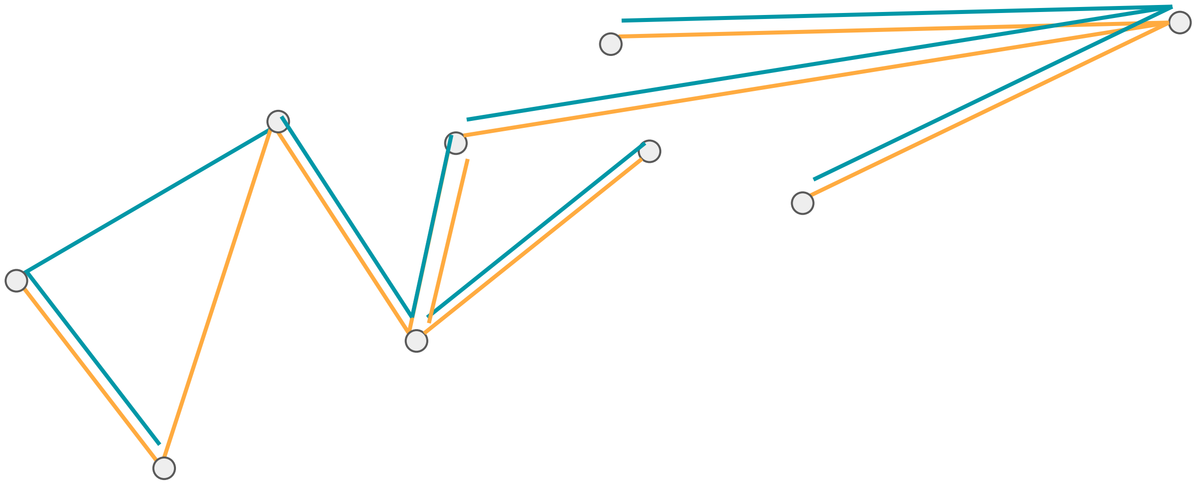

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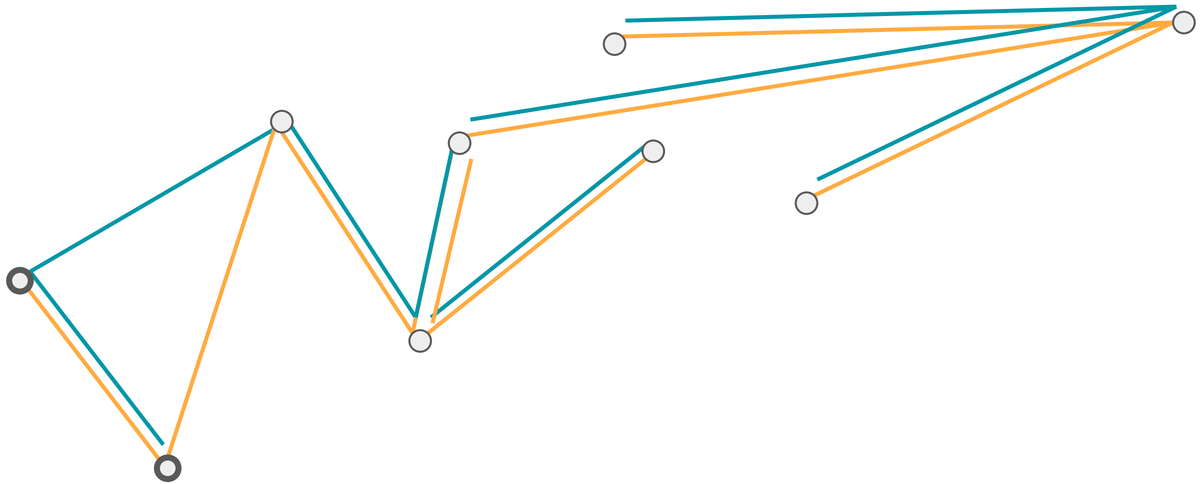

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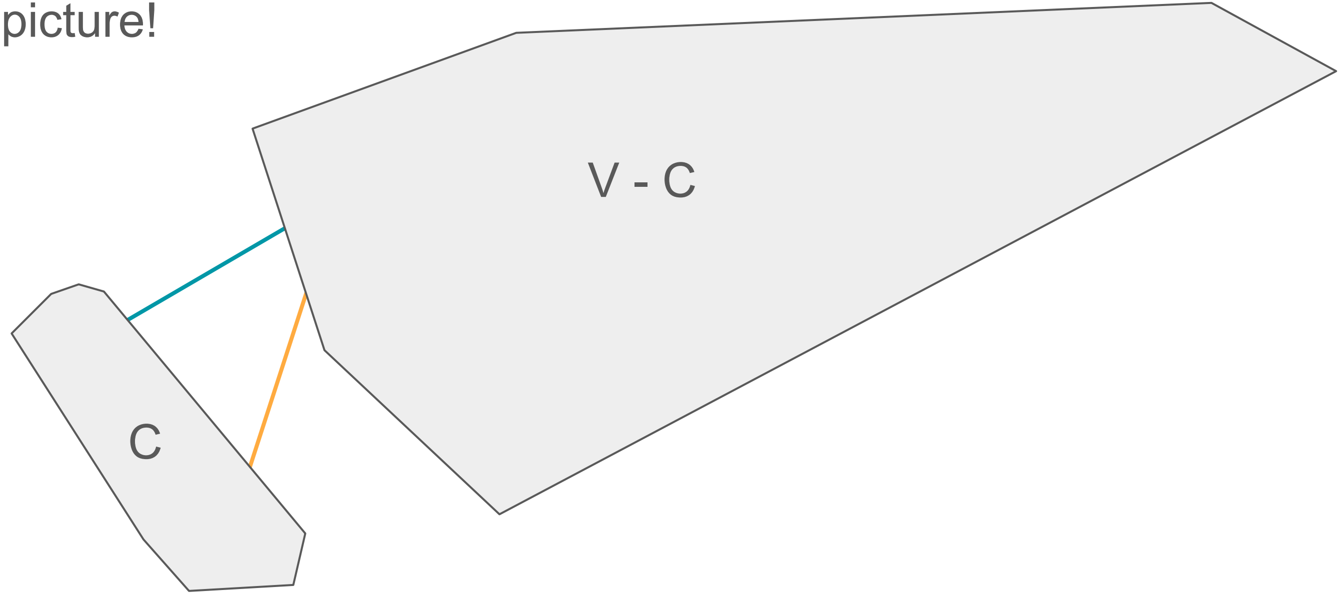

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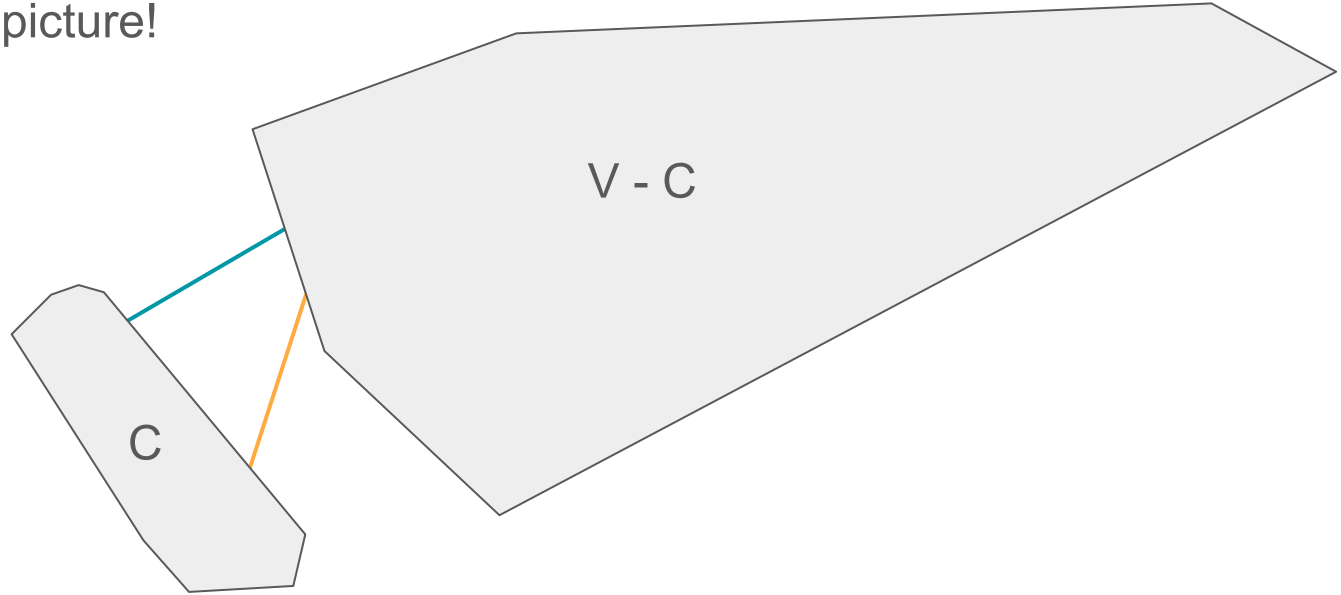

By our assumption, say e_1 is our unique light edge in cut C i.e., $wt(e_1) < wt(e_2)$

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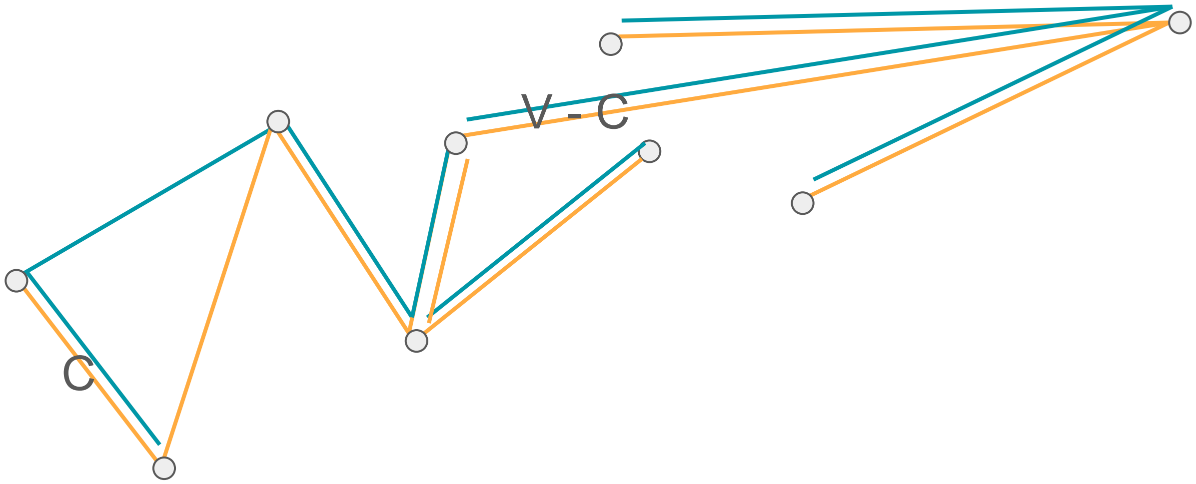

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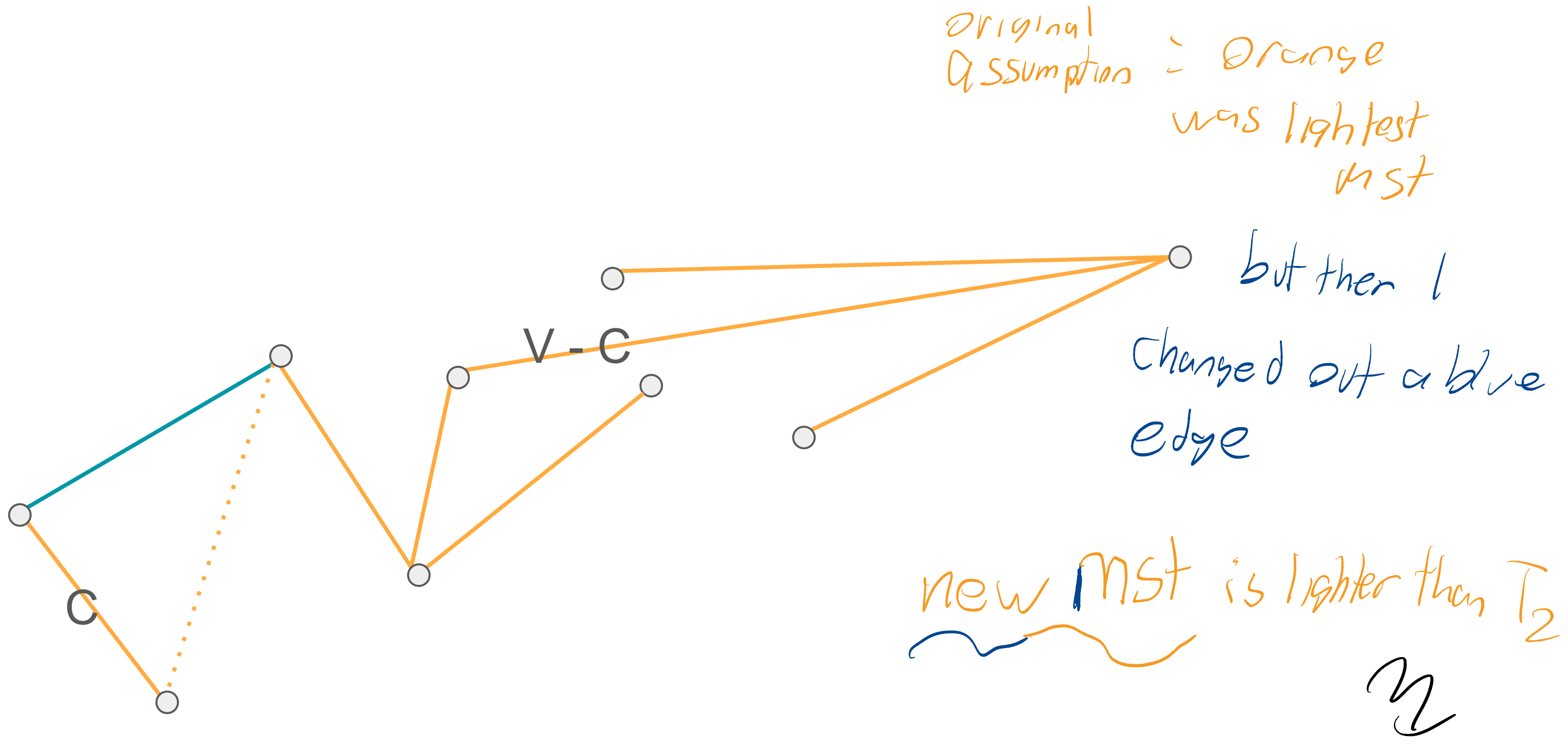

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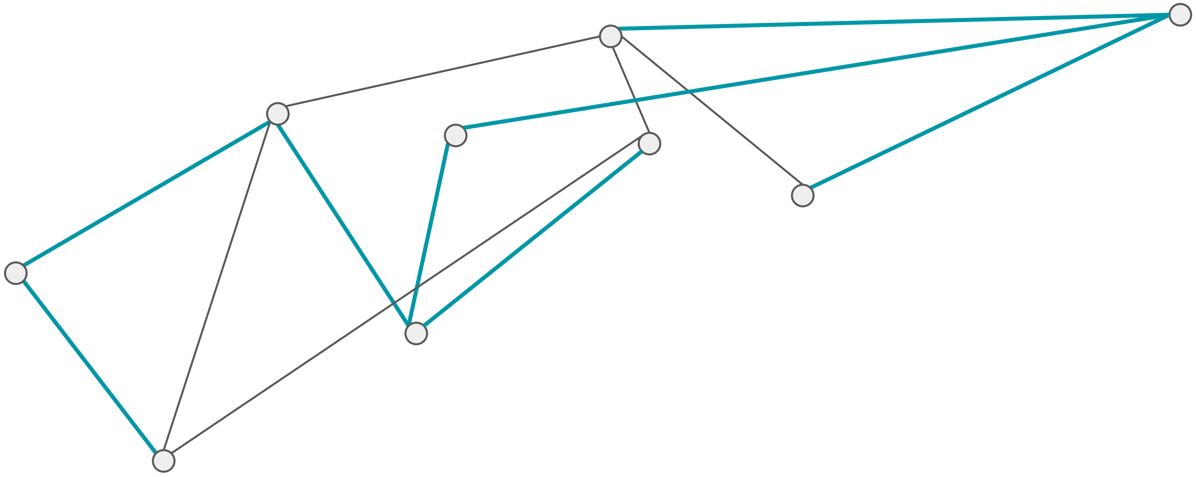
Time for the counter example

3. Let T be an MST of a graph $G = (V, E)$, and let V' be a subset of V. Let T' be the subgraph of T induced by V', and let G' be the subgraph of G induced by V'. Show that if T' is connected, then T' is an MST of G'.

Let this be the graph G and mst T



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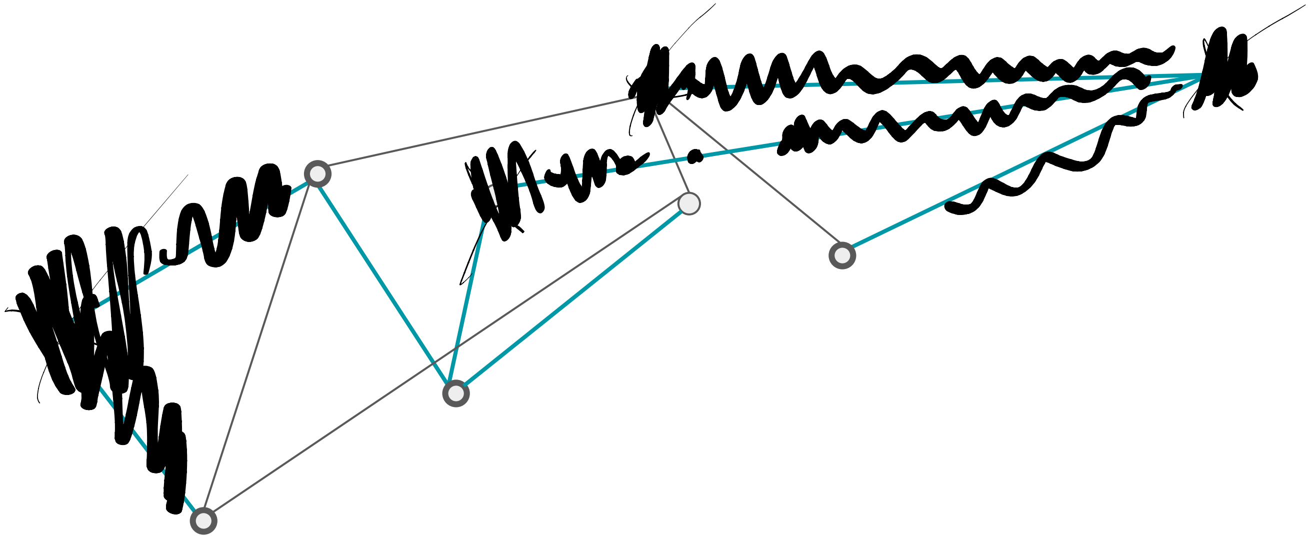

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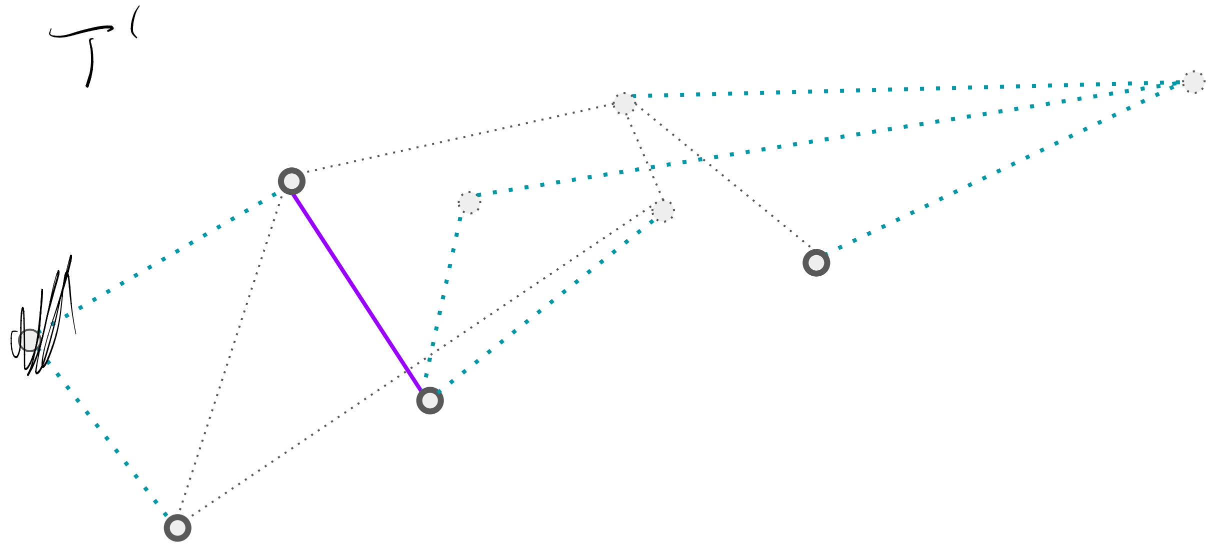

Suppose we define **V'** as follows. This is T', T induced by **V'**

What went wrong? Why isn't a T' MST?

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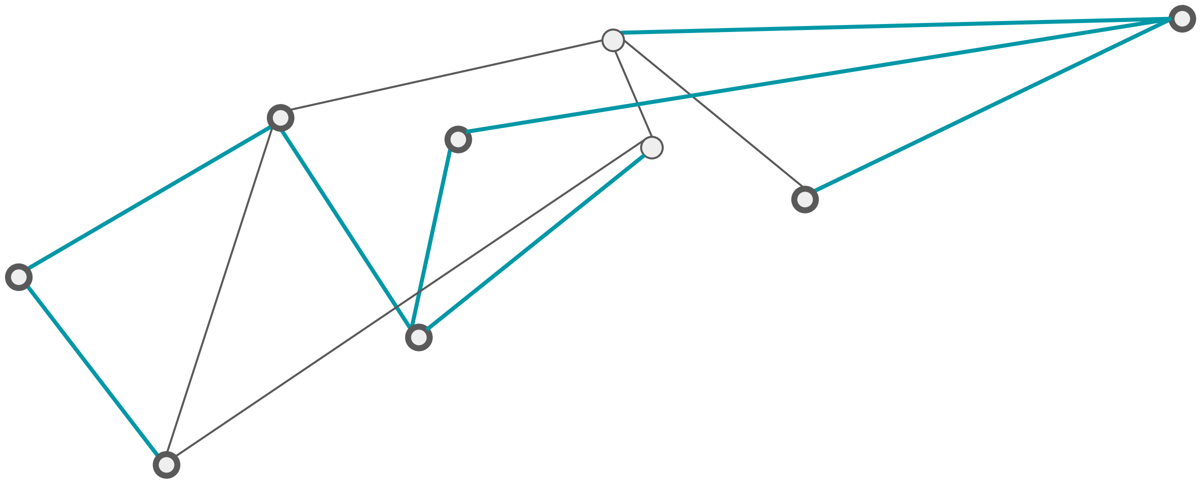

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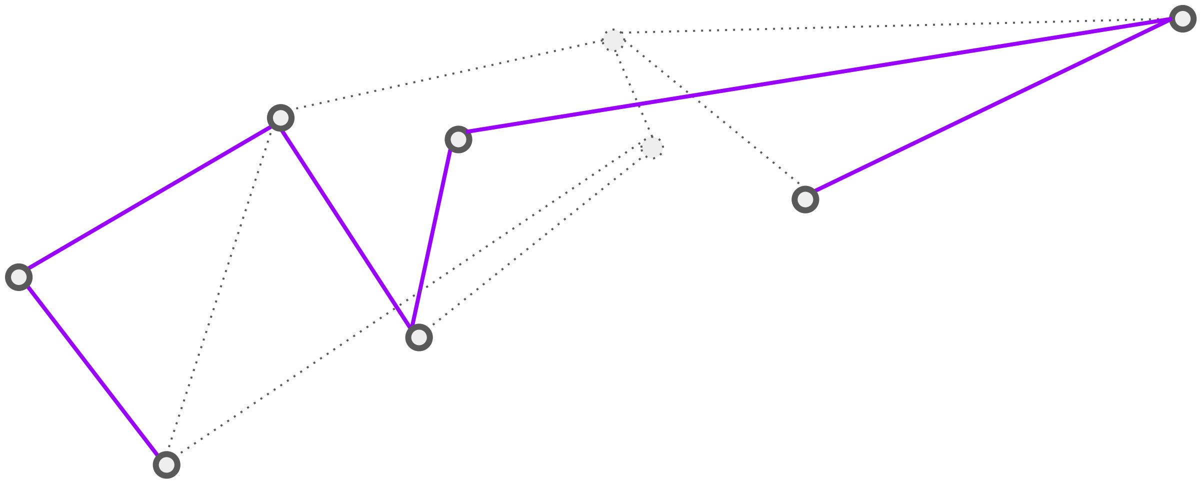

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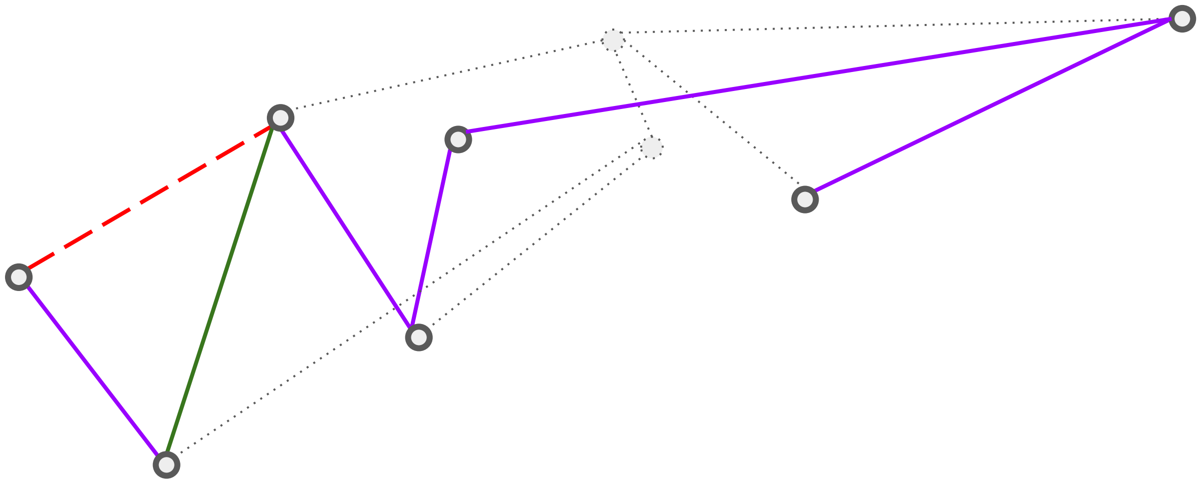

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AFtSoC there is a cheaper tree T'' differing in edges above (added , removed)

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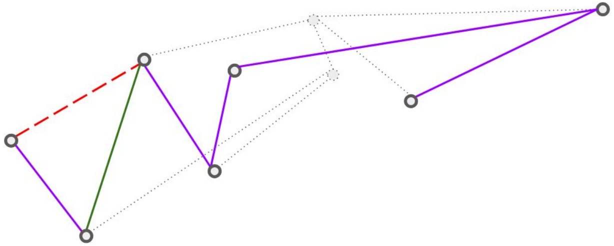

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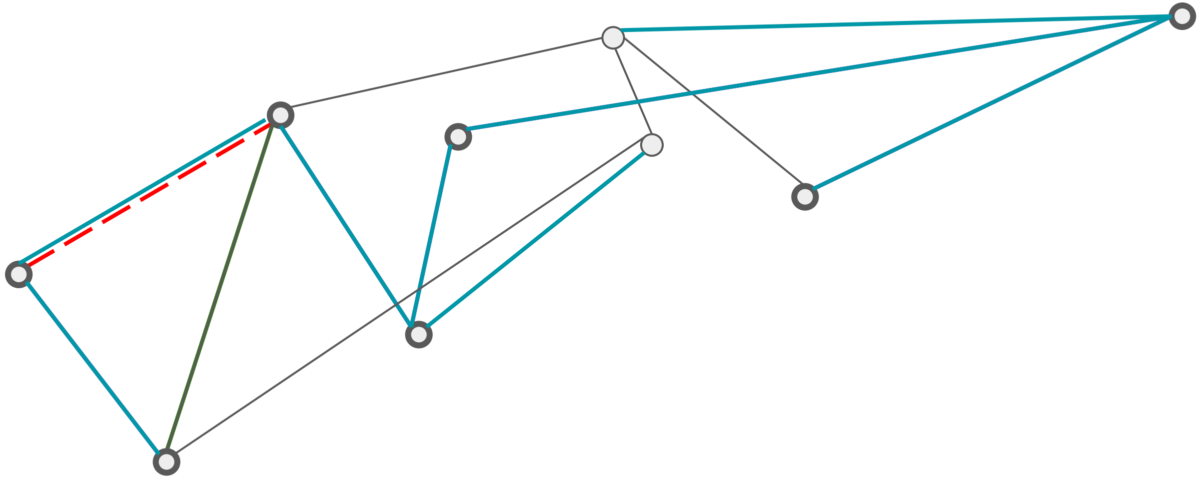

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Back in the original graph we originally had MST T

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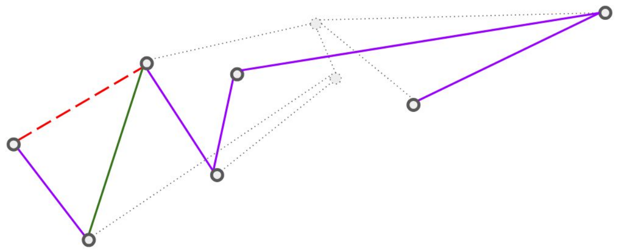

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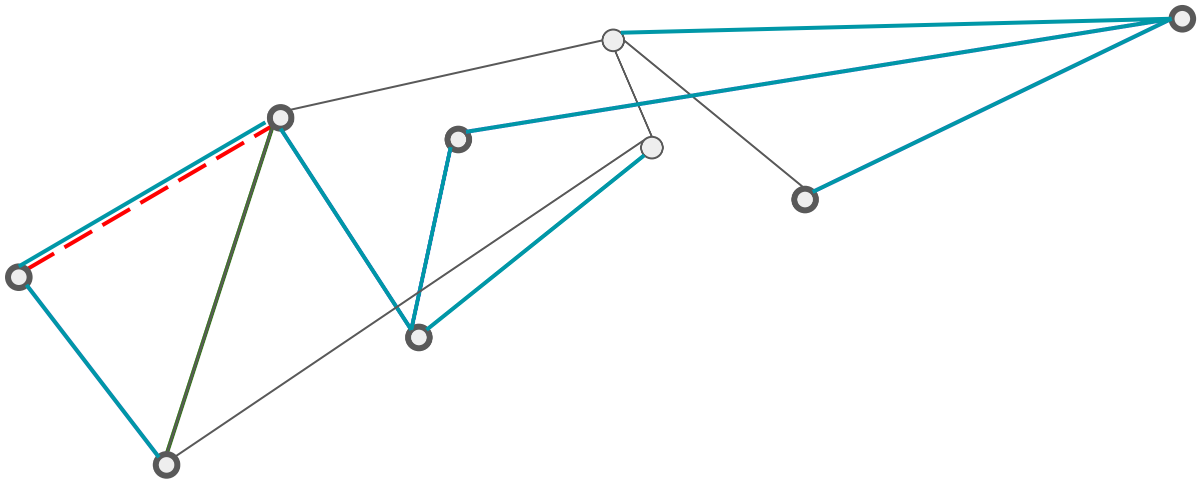

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Removing the red edge and adding the green edge gives us a cheaper tree

Question 2

(Prim's & Kruskal's algorithm)

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Simple Intuition of Prim's algorithm?

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Dijkstra

```
algorithm DijkstraShortestPath(G(V,E), s ∈ V)

   let dist:V → ℤ
   let prev:V → V
   let Q be an empty priority queue

   dist[s] ← 0
   for each v ∈ V do
      if v ≠ s then
         dist[v] ← ∞
      end if
      prev[v] ← -1
      Q.add(dist[v], v)
   end for

   while Q is not empty do
      u ← Q.getMin()
      for each w ∈ V adjacent to u still in Q do
         d ← dist[u] + weight(u, w)
         if d < dist[w] then
            dist[w] ← d
            prev[w] ← u
            Q.set(d, w)
         end if
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   end while

   return dist, prev
end algorithm
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Prim's

Prim's MST

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Pseudocode

```
//Initialize prev, dist
Let dist[v] = current min. edge to v
while pq is not empty:
    Vertex u <- pq.pop()
    for each edge (u,v):
        if wt(u,v) < dist[v]:
            update dist and pq
```

What we can do with an adj matrix

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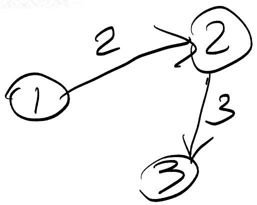
$A[u][v] = wt(u,v)$



$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

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while pq is not empty:

    Vertex u <- pq.pop()

    for each edge (u,v):

        if wt(u,v) < dist[v]:

            update dist and pq

What we can do with an adj matrix

What we cannot do (right away)

Question 2

(Prim's & Kruskal's algorithm)

1. Suppose that we represent the graph $G = (V, E)$ as an adjacency-matrix. Give a simple implementation of Prim's algorithm for this case that runs in $O(|V|^2)$ time.

//Initialize prev, dist

Let dist[v] = current min. edge to v

while pq is not empty:

Vertex u <- pq.pop():

T.add(u)

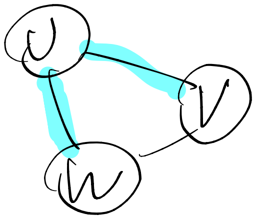
for each edge (u,v):

if wt(u,v) < dist[v]:

Prev[v] = w

Prev[v] = u

update dist and pq



Prims(G,start):

dist = [∞ ...., 0, ∞)

start

//Initialize prev, dist

~~Let T = {start}~~

for i ∈ V − {start}:

Can do this w/ for loop in O(n)

let j be the min. weight neighbor of i

T.add(j)

for k = 1, ..., V such that A[j][k] ≠ 0:

if wt((j, k)) < dist[k]:

dist[k] = wt((j,k))

Prev[k] = j

2. Suppose that all edge weights in a graph are integers in the range from 1 to $|V|$. How fast can you make Kruskal's algorithm run?

Kruskal

- Sort edges by increasing order of their weights // O(?) time
- Run a Union Finding procedure // ~O(|E|) time

With counting sort, Kruskal runs in $O(|E|+|V|)$ time.

The **values** of the edges are bounded by |V|. What's a good sorting algorithm for this?

Faster than $O(|E|\log|E|)$ time?

Counting sort $= O(\text{maxvalue} + |E|)$

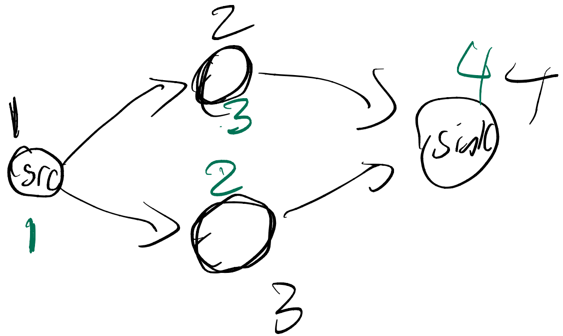
$= O(|V|+|E|)$

Question 3

(Topological Ordering)

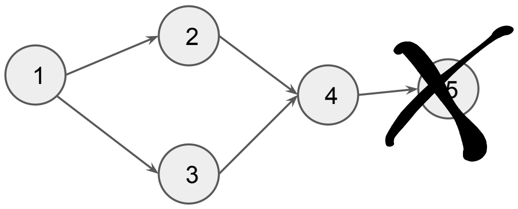
1. Draw a directed acyclic graph $G = (V, E)$ with $|V| = 5$ nodes that has exactly two topological orderings.
2. Prove that G has a topological ordering if and only if G is a DAG.

When do we have two topo orderings?



2. Prove that G has a topological ordering if and only if G is a DAG.

(→) Suppose G has a topo ordering (**WTS:** DAG)



AFTSOC there is a cycle

I can't have a topological ordering.

Fix a topological labeling

∄ a discrepency

(←) Suppose G is a DAG (**WTS:** topo ordering)

- G has a source(s) and a sink(s)
- Assume for all dags G' with n' < n nodes, it has a topo ordering
- Suppose G has n nodes. Remove the sink s.

By IH, there is a topo ordering of G − s. Add s to the end of the order