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CS381 PSO 10: Flow

Problem (Slop-Maxxing). Suppose there is a $n \times n$ grid with the letters S, L, O, and P. Your goal is to form as many *disjoint* copies of the word SLOP as possible from the grid with the following rule: each letter of SLOP must be consecutive, i.e., you can start at any S, move to a neighboring L, move to its neighboring O, and so forth[1]. For example,

S	L	O	P	S
L	S	L	O	P
O	S	O	L	L
S	P	O	S	P
P	S	L	L	O

has solution

S	L	O	P	S
L	*S*	*L*	*O*	*P*
O	S	O	L	L
S	P	O	S	P
P	S	L	L	O

.

Give a polynomial time algorithm to achieve your goal of finding the most SLOP as possible.

Solution: Form a capacitied graph $G = (V, E, C)$ as follows

- **Vertices** V. Add vertices v^{in}, v^{out} for each position in the grid. There are $2n^2$ such vertices. Add a source s and a sink t.
- **Edges** E.
  1. Connect $v^{in} \to v^{out}$ with capacity 1 for each v grid vertices.
  2. Connect u^{out} to v^{in} with capacity 1 if u and v correspond to consecutive letters in SLOP.
  3. Connect $s \to v^{in}$ for all v grid vertices labeled S. Connect $v^{out} \to t$ for all v grid vertices labeled P.

Then, the algorithm is to find and output the max flow from s to t in graph G.

Runtime. Forming the graph takes $O(n^2)$ time. Since the graph has $O(n^2)$ vertices and $O(n^2)$ edges, this takes $O(n^6)$ time with Edmonds Karp.

Correctness. We want to show that $\mathcal{S}$ is a set of disjoint slop iff there exists a flow in the graph $G = (V, E)$ with value $|\mathcal{S}|$.

($\to$) If $\mathcal{S}$ is a set of disjoint slop, then

$$\mathcal{S} = \{(s_i, l_i, o_i, p_i) : i \in [|\mathcal{S}|]\}$$

where (1) s_i, l_i, o_i, p_i are the coordinates of some consecutive S, L, O, P respectively, and (2) for $i \neq j$, $s_i \neq s_j, l_i \neq l_j$, $o_i \neq o_j$, and $p_i \neq p_j$. Then, observe that in the graph G, for each $i \in [|\mathcal{S}|]$,

$$s \to (s_{i,in} \to s_{i,out}) \to (l_{i,in} \to l_{i,out}) \to (o_{i,in} \to o_{i,out}) \to (p_{i,in} \to p_{i,out}) \to t$$

corresponds to a unit of flow in the graph G. Hence, there exists a flow of size $|\mathcal{S}|$.

[1] Here, a letter Y is neighboring X if you can move up, down, left, or right once from X to get to Y, and vice versa.

($\leftarrow$) Suppose there exists some flow with value k. By the **first type of edges ($u_{in} \rightarrow u_{out}$) of capacity 1, the flows are in vertex disjoint paths from $s \rightarrow \cdots \rightarrow t$.** Hence, by the graph structure, each unit of flow corresponds to exactly one disjoint path

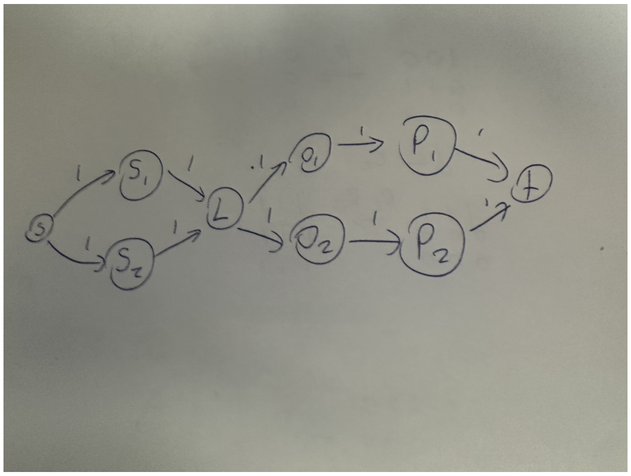
$$p_i := s \rightarrow (s_{i,in} \rightarrow s_{i,out}) \rightarrow (l_{i,in} \rightarrow l_{i,out}) \rightarrow (o_{i,in} \rightarrow o_{i,out}) \rightarrow (p_{i,in} \rightarrow p_{i,out}) \rightarrow t.$$

In other words, there exists a bijection f between $[k]$ and disjoint paths $p_1, \dots, p_k$.[2] By construction and the vertex disjointness, we can correspond this to exactly k disjoint SLOPs in the grid.

Justin says: Remember the graph construction presented in PSO (no in/out split)? I mentioned that you would have difficulty proving the backwards direction, and here is why: the incorrect graph allows for more than 1 flow through some vertex v. This corresponds to a grid cell that is used more than once, so this is not a disjoint slop solution! Take for example,

O	S_2	O	O
S_1	L	O_1	P_1
O	O_2	O	O
O	P_2	O	O

There is only 1 instance of (disjoint) slop (either $S_1LO_1P_1$ or $S_2LO_2P_2$). If we did not have



the in/out split, the graph we form (pictured above) has flow 2, when we wanted a flow of 1.

Justin says: Where does this show up in the proof? Remember the bolded-part in the backward direction where we said each flow corresponded to a vertex disjoint path? Here that is clearly not the case since L is used twice! The flow function (no longer bijective) mentioned looks like

$$f : 1 \mapsto sS_1LO_1P_1t \quad ; \quad 2 \mapsto sS_2LO_2P_2t$$

so we can no longer get a disjoint SLOP set equal to the flow of 2.

[2] Think of the input as choosing the i'th path of flow.