7.1.1 Let $G = (V, E)$ be an unweighted directed graph with n vertices and m edges. For two vertices s and t, compute the length fo the shortest walk from s to t with an even number of edges. **Give an algorithm and analyze it.**

We proceed with an auxiliary graph-type solution.

Graph Construction. Construct graph $G' = (V', E')$ as follows:

1. Make two copies of vertices V_e and V_o defined as
$$V_e = \{v_{e,i} : v_i \in V\} \text{ and}$$
$$V_o = \{v_{o,i} : v_i \in V\}.$$
2. Construct edges E' as
$$E' = \{(u_e, v_o), (u_o, v_e) : (u, v) \in E\}.$$
3. Construct $V' = V_e \cup V_o$.

The algorithm. On input (s, t):

1. Construct graph G'.
2. Run $\texttt{BFS}(s_e, t_e)$ on auxiliary graph G' to get path
$$s_e \to x \to y \to \cdots \to t_e.$$
3. Output path
$$s \to x \to y \to \cdots \to t$$
in the original graph G.

Runtime. Constructing graph G' takes $O(n + m)$ time. Since $|V'| = O(n)$ and $|E'| = O(m)$, running $\texttt{BFS}(s_e, t_e)$ takes $O(n+m)$ time. Lastly, since outputting the corresponding path in the original graph takes at most $O(n + m)$ time, the entire algorithm takes $O(n + m)$ time.

Correctness. Calling $\texttt{BFS}(s_e, t_e)$ gives the shortest path from $s_e \rightsquigarrow t_e$ in G'. We want to show that our mapped path in step 3 correctly computes the shortest even path.

WTS: there is a one-to-one correspondence between all *even* paths from $s \rightsquigarrow t$ in G and all paths from $s_e \rightsquigarrow t_e$ in G'. That is, there is a bijection between paths

$$(s \to x \to \cdots \to t) \text{ in } G, \text{ and paths } (s_e \to x \to \cdots \to t_e) \text{ in } G'.$$

($\to$) Consider path $(s \to x \to y \to \cdots \to t)$ in G. This directly corresponds to path $(s_e \to x_o \to y_e \to \cdots \to t_e)$ in G', where each odd step vertex x is mapped to x_o and each even step vertex y is mapped to y_e. (This path exists by construction of G')

($\leftarrow$) Consider path $(s_e \to x_o \to \cdots \to t_e)$ in G'. The corresponding path in G is $(s \to x \to \cdots \to t)$.

What is left to show is that this path is even. Observe that the length of path $(s_e \to x_o \to \cdots \to t_e)$ is even, since the only way to start at an even vertex s_e and end at an even vertex t_e is to take an even number of steps. Thus, the mapped path $(s \to x \to \cdots \to t)$ in G is an even length path as desired.

Since the one to one mapping between even length paths in G from $s \rightsquigarrow t$ and paths in G' from $s_e \rightsquigarrow t_e$ is exactly the mapping we do in step 3 of our algorithm, correctness follows.