

Lowkenuinely Non-Linear. Given n distinct integers $z_1, \dots, z_n$, we want to find an integer x that minimizes the following quantity:

$$\sum_{i=1}^n (x - z_i)^2.$$

Give an $O(n)$ time algorithm for finding such a x and prove correctness of your algorithm.

Solution. First, we show that the minimizing point for function $f(x) = \sum_{i=1}^n (x - z_i)^2$ is the mean $\bar{z} := (z_1 + \dots + z_n)/n$. Then, computing the mean takes $O(n)$ time via an elementary linear scan, and we are done.

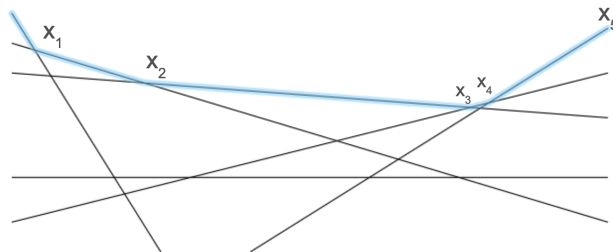
Indeed, since each summand is convex, the entire sum is convex, and the minimum occurs at the zero of the derivative $f'(x) = 0$. Computing the derivative,

$$f'(x) = \sum_{i=1}^n 2(x - z_i) = 0 \iff nx = \sum_{i=1}^n z_i \iff x = \frac{1}{n} \sum_{i=1}^n z_i = \bar{z}.$$

Nobody Tell Kent Quanrud That I Stole Another One of His Problems. Let L be a set of n lines of the plane, where the i 'th line is of the form. $y_i(x) = a_i x + b_i$ for $a_i, b_i \in \mathbb{R}$. The *upper envelope* is the function

$$U_L(x) = \max_{i \in [n]} y_i(x).$$

Observe that U_L is piecewise-linear and convex. We represent the upper envelope as a list of m breakpoints $(x_1, \ell_1), \dots, (x_m, \ell_m)$ where $x_1 < \dots < x_m$ are the x -coordinates where the topmost line changes, and ℓ_i is the line achieving the maximum for $x \in [x_i, x_{i+1})$.



Design and analyze a recursive algorithm computing the upper envelope of n lines¹.

¹Hint: n lines induce $\binom{n}{2}$ points of intersection, but how many breakpoints can the upper envelope have?