PSO 1

Recurrences, divide and conquer

https://justin-zhang.com/teaching/CS381

Exercise 1.1. Use recursion trees to solve the following recurrences.

1. $T(n) = 3T(n/3) + 6n$ and $T(1) = 2$.
2. $T(n) = 2T(n/3) + 4n$ and $T(1) = 7$.
3. $T(n) = 4T(n/3) + n$ and $T(1) = 11$.

Exercise 1.3. Problems 1–5 describe a search problem on a nonempty array $A[1..n]$ of comparable elements.[7] For each of these problems, design a recursive algorithm that solves the search problem as fast as possible.[8] Briefly analyze the running time of each algorithm.[9]

1. Let $A[1..n]$ be in a *rotated* sorted order. That is, there exists an index $i \in [n]$, called the *rotation index*, such that the concatenation of $A[i..n]$ and $A[1..i-1]$ is sorted in increasing order. (One can think of $A[1..n]$ as being sorted initially, and then rotated cyclically to the right by $i-1$ slots). The goal is to compute the unknown rotation index i. For simplicity, you may assume all the elements are distinct.

Exercise 1.7. Let $A[1..n] \in \mathbb{R}^n$ be an array of n numbers. An *inversion* is a pair of numbers out of increasing order; more precisely, a pair of indices $i, j \in [n]$ such that $i < j$ and $A[i] > A[j]$. Design and analyze an algorithm (as fast as possible) for counting the number of inversions in A.

. Give a linear time algorithm taking as input a tree and returning a centroid. Your proof of correctness doubles as a proof that every tree has a centroid.

Exercise 2.2. For each of the recursive specifications below, design a recursive algorithm implementing the specification. (No proof or analysis is needed.)[5]

1. `subset-sum`($x_1, \ldots, x_n$, T): Given n integers $x_1, \ldots, x_n \in \mathbb{Z}$, and an additional integer T, returns `true` if there is a subset of indices $S \subseteq [n]$ such that $\sum_{i \in S} x_i = T$, and `false` otherwise.

1. $T(n) = 3T(n/3) + 6n$ and $T(1) = 2$.

Recall the steps of a recursion tree

1. Draw out the tree
2. Find the cost at the ith level and the number of levels
3. Derive the sum and closed form

1. $T(n) = 3T(n/3) + 6n$ and $T(1) = 2$.

Each node tells us the *non-recursive* work done.
Start with the root as T(n)

Have a branch for each recursive call

T(n): 6n

1. Draw out the tree
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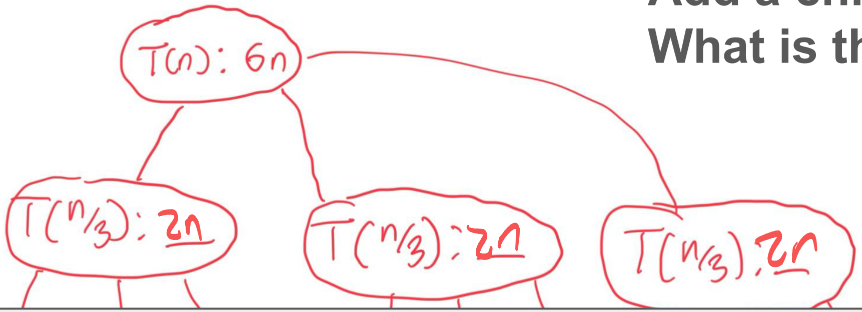
1. $T(n) = 3T(n/3) + 6n$ and $T(1) = 2$.

Each node tells us the *non-recursive* work done.
Add a child for each recursive call
What is the non-rec. work done in each child?



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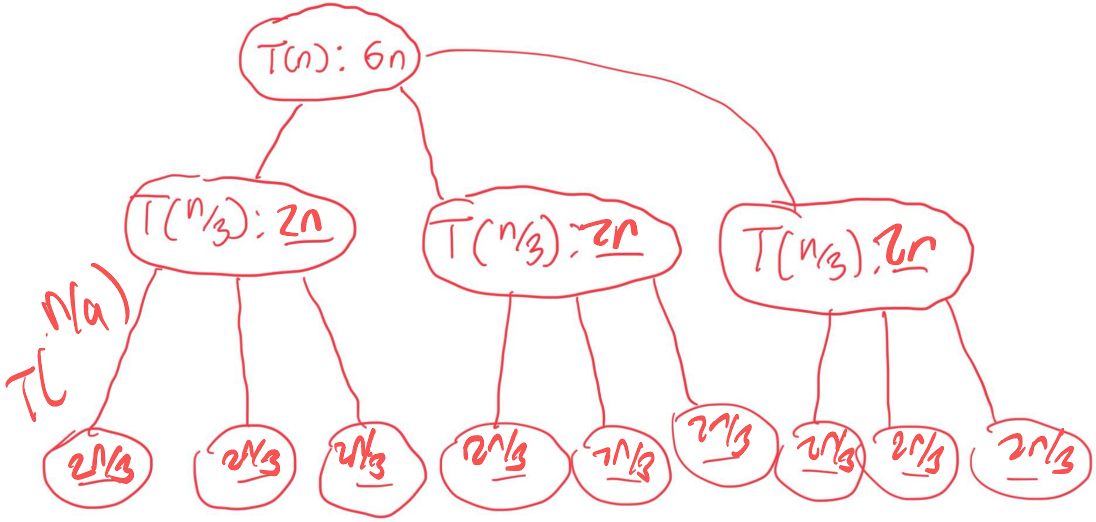

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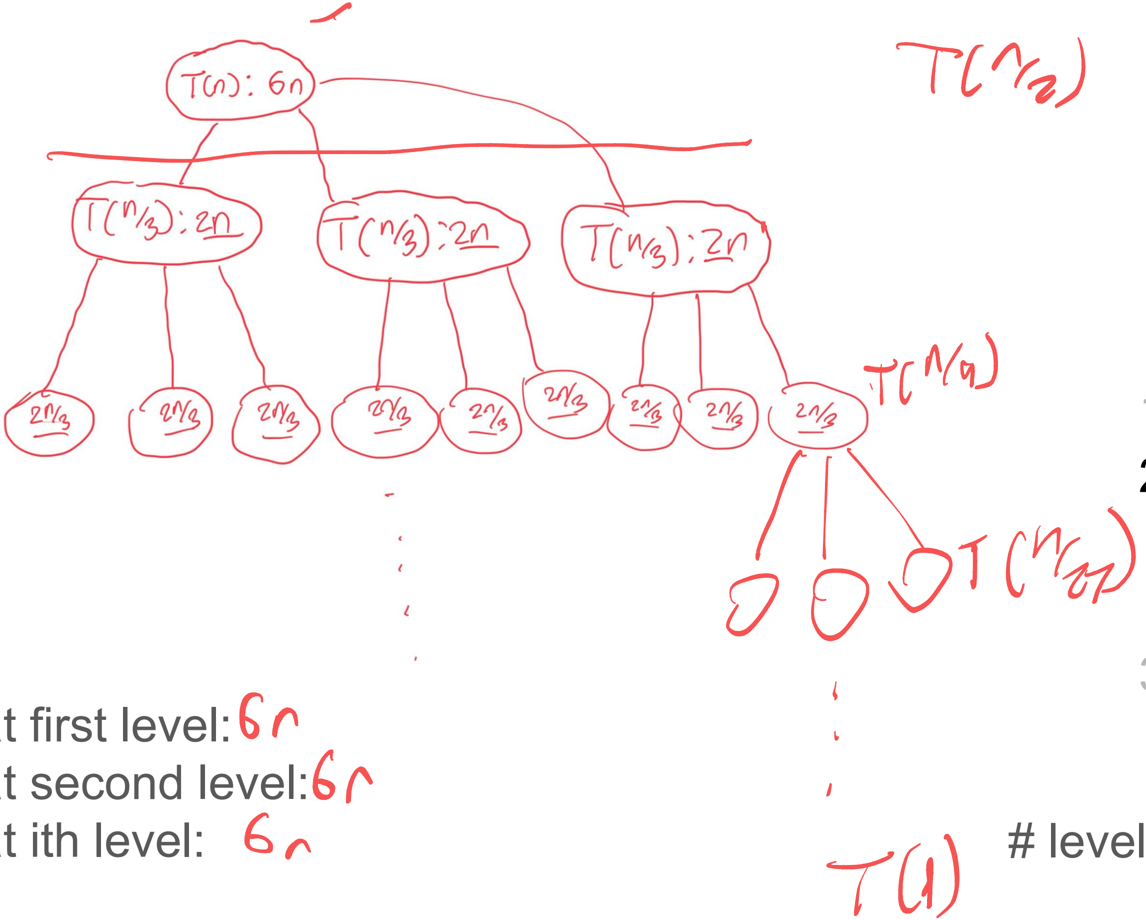
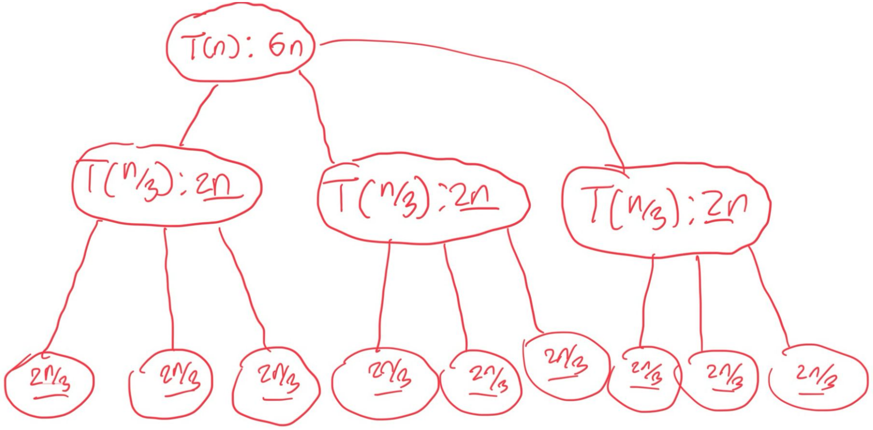


1. Draw out the tree
2. Find the cost at the ith level and the number of levels
3. Derive the sum and closed form

Cost at first level: $6n$
Cost at second level: $6n$
Cost at ith level: $6n$

\# levels: $\log_3 n = O(\log n)$

1. $T(n) = 3T(n/3) + 6n$ and $T(1) = 2$.



C_i cost @ ith level: $6n$

N number of levels: $\log n$

Solve sum:

$$\sum_{i=1}^{N} c_i = \sum_{i=1}^{\log n} 6n = O(n \log n)$$

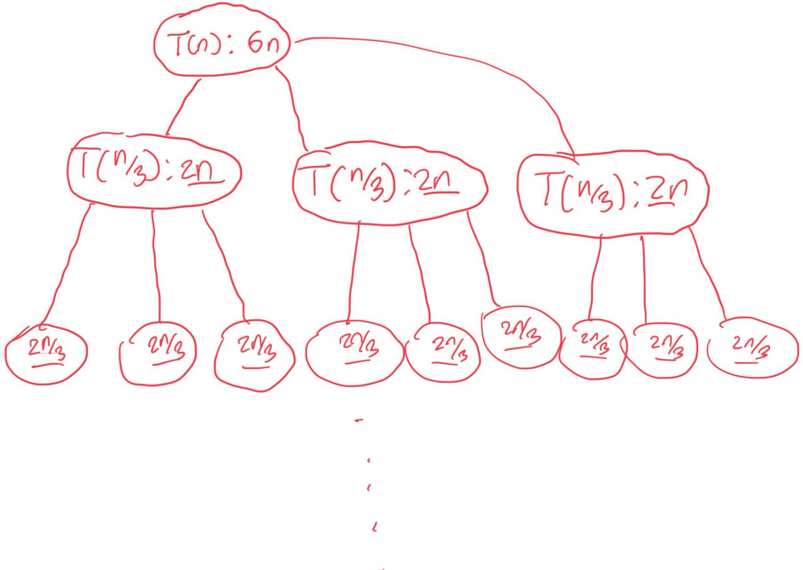
1. Draw out the tree
2. Find the cost at the ith level and the number of levels
3. Derive the sum and closed form

(big-O is fine)

Aside: this is the "merge sort" recurrence

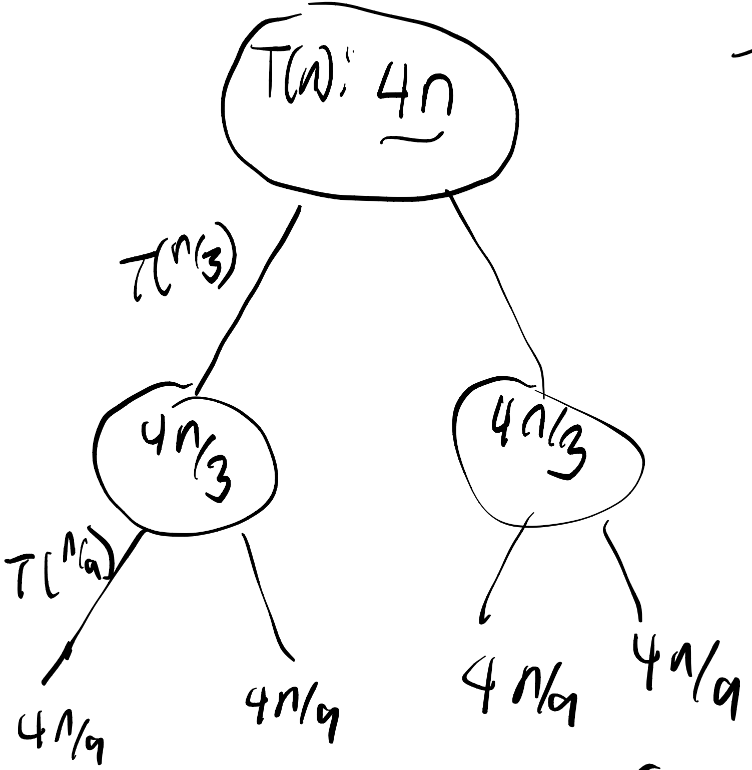
For any constant c and C,

$$T(n) = cT(n / c) + Cn \in O(n\log n)$$



Each level will sum to Cn!
(Here C was 6)

2. $T(n) = 2T(n/3) + 4n$ and $T(1) = 7$.



Cost

$4n = (2^0)4n$

$8n/3 = (2^1)(4n/3)$

$16n/9 = (2^2)(4n/9) = \frac{(2^2)(4n)}{(3^2)}$

$= (2/3)^2 \times (4n)$

$C_i = (2/3)^i \times (4n)$

#levels: log n

$\sum_{i=1}^{\log n} (2/3)^i (4n)$

$\leq 4n \sum_{i=0}^{\infty} (2/3)^i = \boxed{O(n)}$

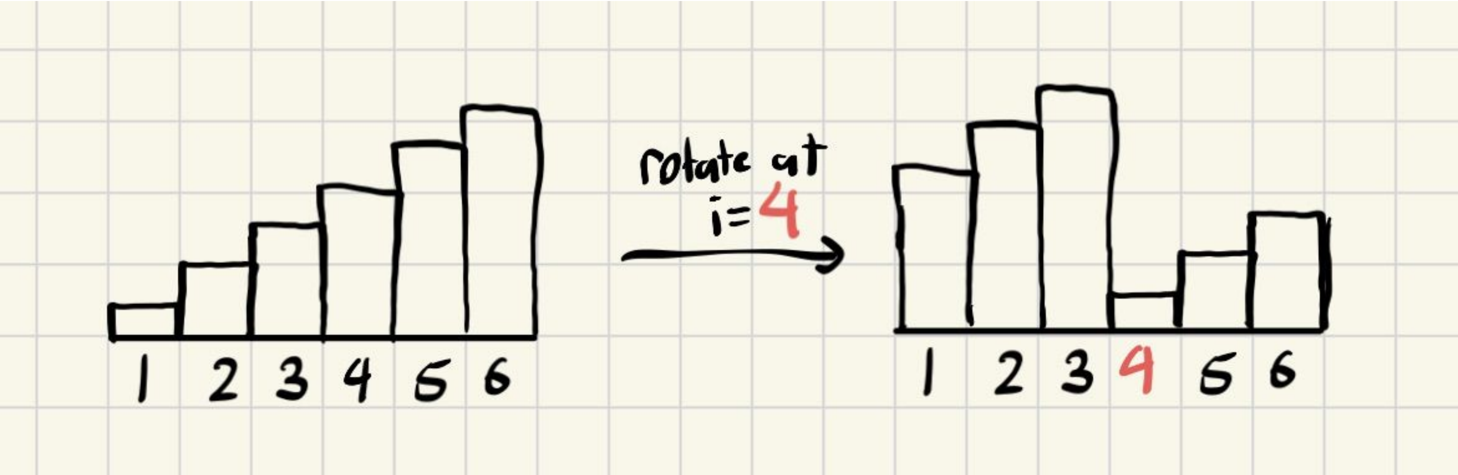
1. Draw out the tree
2. Find the cost at the ith level and the number of levels
3. Derive the sum and closed form

3. $T(n) = 4T(n/3) + n$ and $T(1) = 11$.

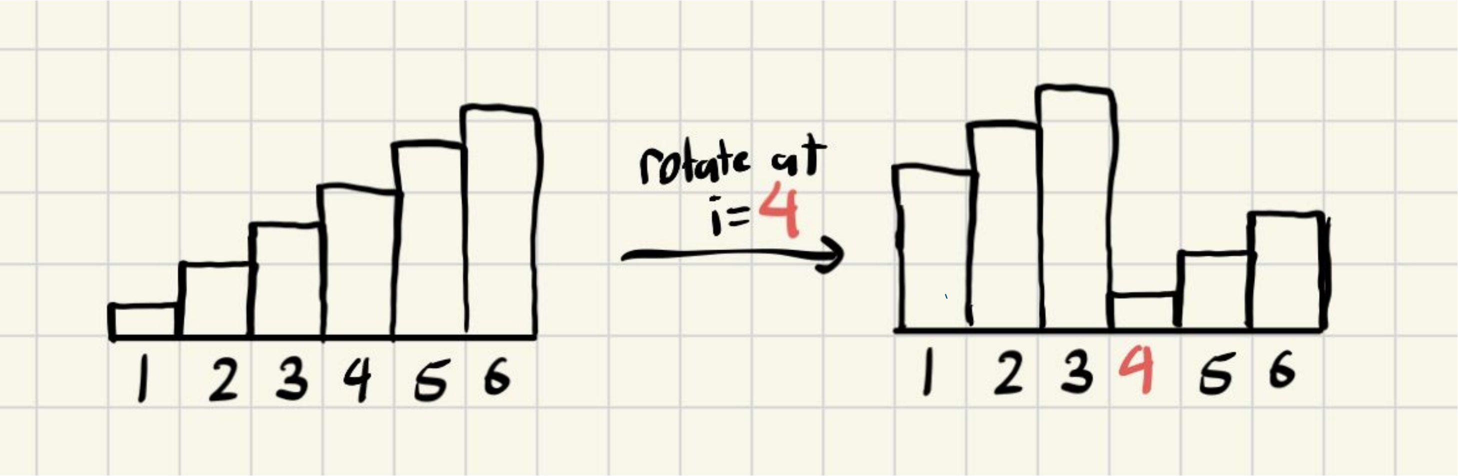
Exercise for you :)

1. Draw out the tree
2. Find the cost at the ith level and the number of levels
3. Derive the sum and closed form

1. Let $A[1..n]$ be in a *rotated* sorted order. That is, there exists an index $i \in [n]$, called the *rotation index*, such that the concatenation of $A[i..n]$ and $A[1..i-1]$ is sorted in increasing order. (One can think of $A[1..n]$ as being sorted initially, and then rotated cyclically to the right by $i - 1$ slots). The goal is to compute the unknown rotation index i. For simplicity, you may assume all the elements are distinct.



Goal: find index i given rotated array

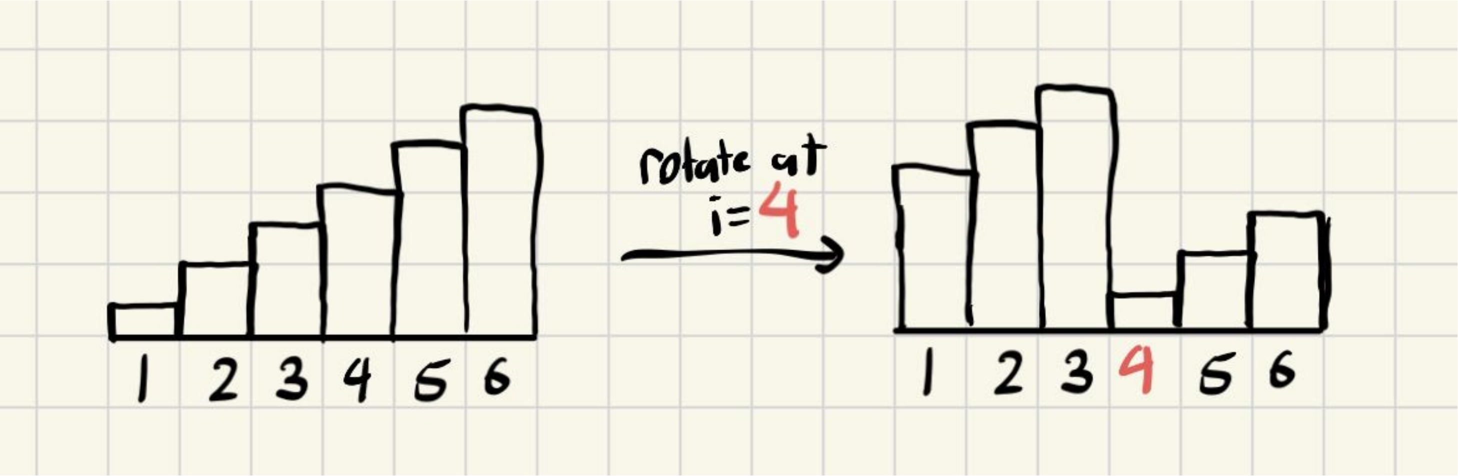


Any ideas (maybe from lecture)? binary search

O(logn)

O(n) easy

Goal: find index i given rotated array



Any ideas (maybe from lecture)? **Binary search type question**

More generally, a recursive algorithm!

Steps to writing a recursive algorithm

1. Write the recursive spec
   a. What is the input?
   b. What is the output?
2. Use the recursive algorithm on smaller parts
3. Combine them appropriately

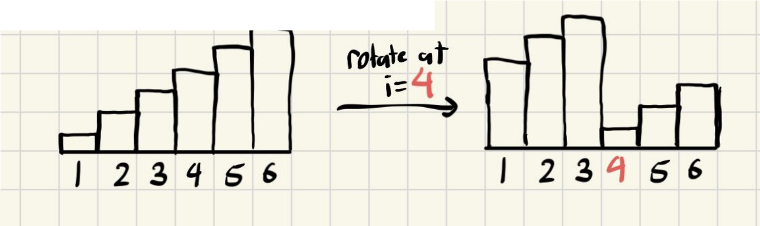
Example 1 : spec of merge sort

1) input

`merge-sort`($A[1..n]$): Given an array $A[1..n]$ of comparable elements, returns an array consisting of the elements of A sorted in increasing order.

2) output

1. Let $A[1..n]$ be in a *rotated* sorted order. That is, there exists an index $i \in [n]$, called the *rotation index*, such that the concatenation of $A[i..n]$ and $A[1..i-1]$ is sorted in increasing order. (One can think of $A[1..n]$ as being sorted initially, and then rotated cyclically to the right by $i-1$ slots). The goal is to compute the unknown rotation index i. For simplicity, you may assume all the elements are distinct.



find-rot(A[1,...,n]): Given rotated-sorted array A,

(precondition)

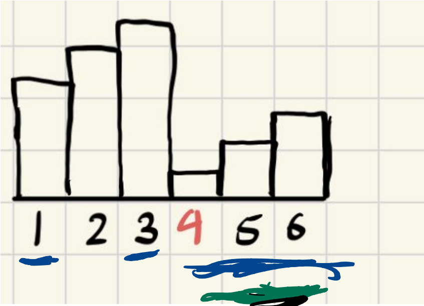
return rotation index i.

(Post condition)

find-rot(A[1,...,n]): Given rotated-sorted (pre condition) array A,

return rotation index i (Post condition).

1. Write the recursive spec
2. **Use the recursive algorithm on smaller parts**
3. Combine them appropriately

find-rot ()

if A[1] < A[m]:

// rotation index is in the right half.

find_rot(A[m+1, n])

m = 3 ≠ 1

We want to binary search for index 4

(m) Middle index = 6 / 2 = 3

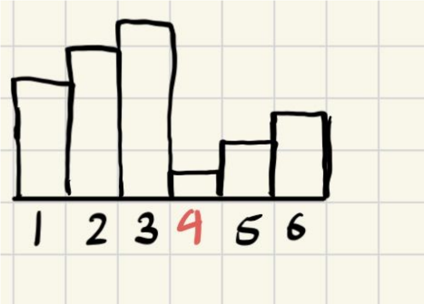
How do we choose between recursing into left half or right half?
I.e., which half has the rotated index?

(if not in right half return find-rot(A,...,m)) find-rot(A[1,...,n]): Given rotated-sorted (pre condition) array A,

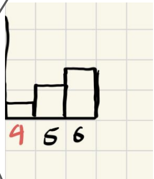
return rotation index i (post condition).

1. Write the recursive spec
2. **Use the recursive algorithm on smaller parts**
3. Combine them appropriately

find-rot



find-rot



1. subset-sum($\overbrace{x_1, \ldots, x_n}^{X}$, T): Given n integers $x_1, \ldots, x_n \in \mathbb{Z}$, and an additional integer T, returns true if there is a subset of indices $S \subseteq [n]$ such that $\sum_{i \in S} x_i = T$, and false otherwise.

$[1, 2, 3, 4], 5$

all subsets : 2^X

I can split 2^X in half as:

1) Subsets with x_1 (recursively listing all subsets)

Subsets without x_1

$f(X = \{x_1, \ldots, x_n\})$:

If $X = \emptyset$: return $\{\emptyset\}$

$S = f(\{x_2, \ldots, x_n\})$

return $\{\{x_1\} \cup s, s\} : s \in S\}$.

1. `subset-sum`($x_1, \ldots, x_n$, T): Given n integers $x_1, \ldots, x_n \in \mathbb{Z}$, and an additional integer T, returns **true** if there is a subset of indices $S \subseteq [n]$ such that $\sum_{i \in S} x_i = T$, and **false** otherwise.

def $SS(x_1, \ldots, x_n, T)$:

if $T = 0$: // $\emptyset$
return true.

if $n = 0$:
return false.

return

$\underbrace{SS(\cancel{x_1}, \ldots, x_n, T - x_i)}_{\text{including } x_i}$ or $\underbrace{SS(\cancel{x_1}, x_2, \ldots, x_n, T)}_{\text{excluding } x_i}$