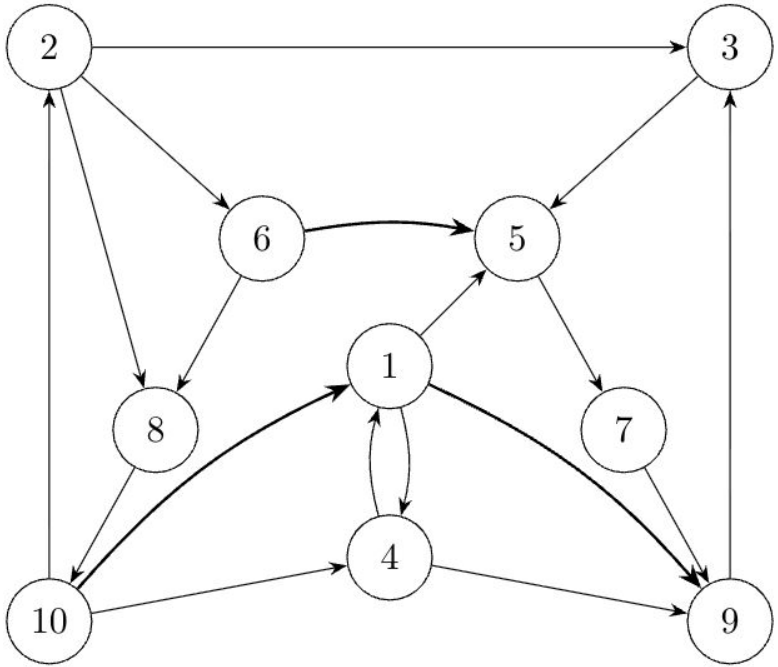
PSO 11

Kosaraju, Dijkstra, Bellman Ford

Question 1

(Kosaraju's algorithm)

(a) Run phase 1 of Kosaraju's algorithm and show the L stack at the end of phase 1. (Note: You should assume that that we loop through nodes in numerical order (ascending) and that each adjacency list are also sorted in ascending order e.g., the adjacency list for node 1 is $(4, 5, 9)$)



(b) Run phase 2 of Kosaraju's algorithm and list the strongly connected components (in topological order).

Question 2

(Dijkstra's algorithm)

1. Give a simple example of a directed graph with negative-weighted edges for which Dijkstra's algorithm produces an incorrect answer.

2. Given a weighted, directed graph $G = (V, E)$ in which edges that leave the source vertex s may have negative weights, but all other edge weights are non-negative, and there are no negative-weighted cycles. Can the Dijkstra's algorithm correctly find all the shortest paths from s in this graph?

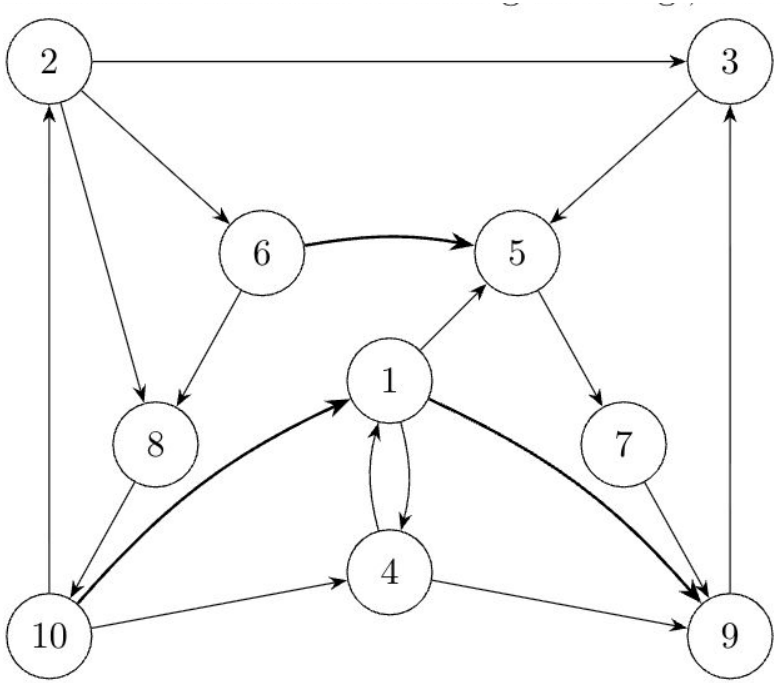
3. Your classmate claims that Dijkstra's algorithm relaxes the edges of every shortest path in the graph in the order in which they appear on the path. Show that him/her is mistaken by constructing a directed graph for which the Dijkstra's algorithm could relax the edges of a shortest path out of order.

Hint: The shortest path between two vertices in the graph is not necessarily unique.

Question 3

(Bellman-Ford algorithm)

1. Why does the Bellman-Ford algorithm only require $|V| - 1$ passes?
2. Why will the last pass ($|V| - 1$) through the edges will determine if there are any negative weight cycles or not?



Kosaraju's Algorithm

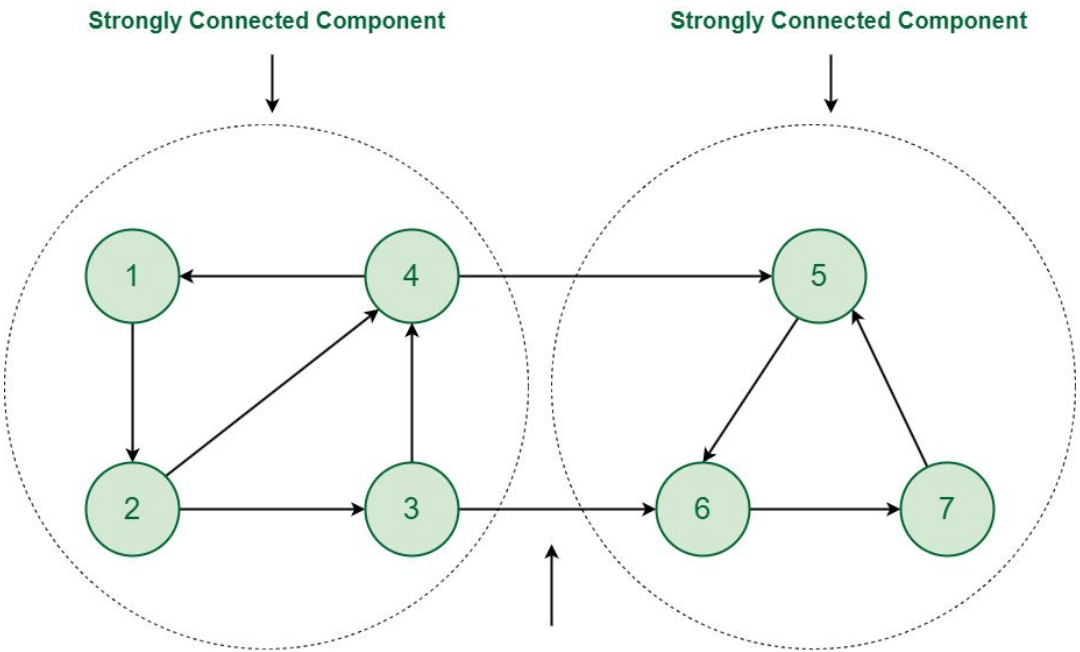
outputs:

How does it work:

Phase 1:

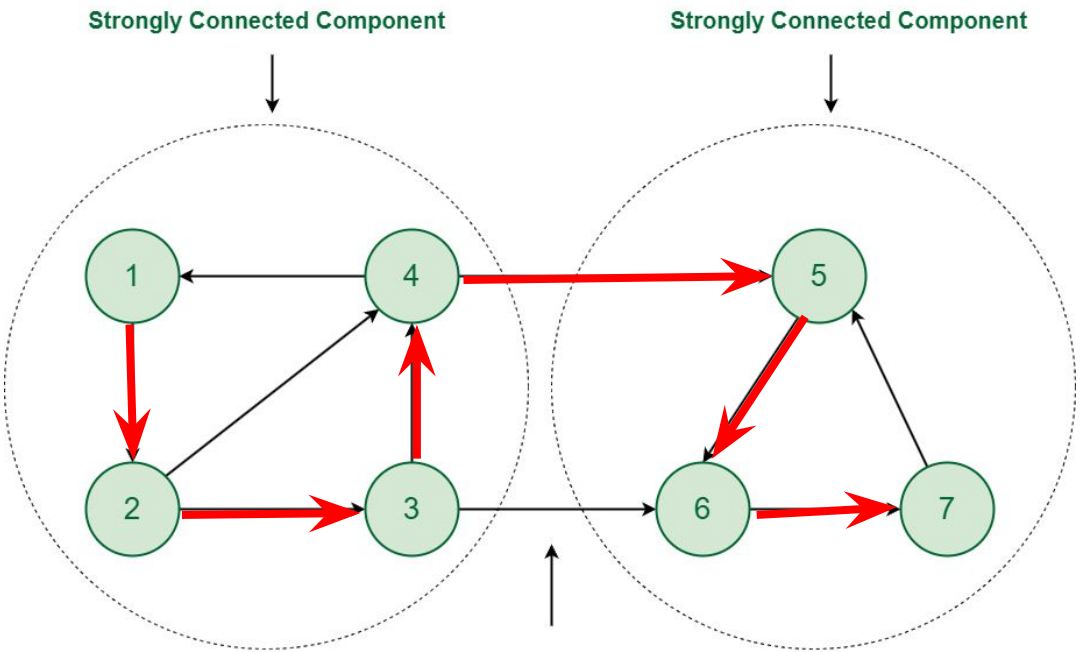
Phase 2:

Intuition of Kosaraju's (Phase 1)



Phase 1: Find all *unidirectional* connected components with DFS

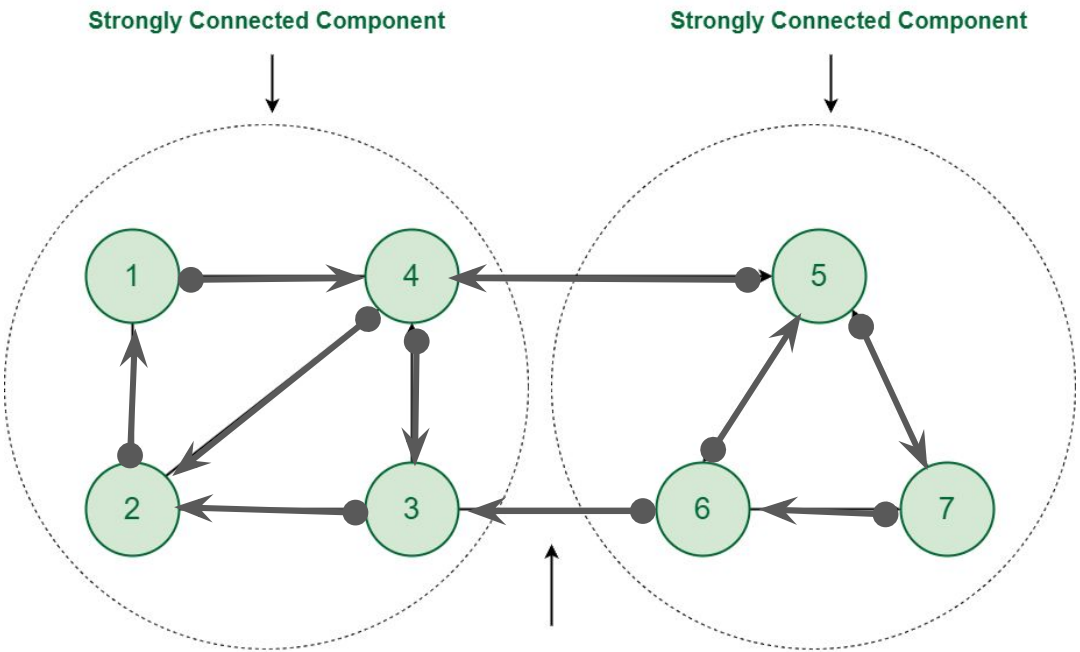
Intuition of Kosaraju's (Phase 1)



Phase 1: Find all *unidirectional* connected components with DFS

Intuition of Kosaraju's (Phase 2)

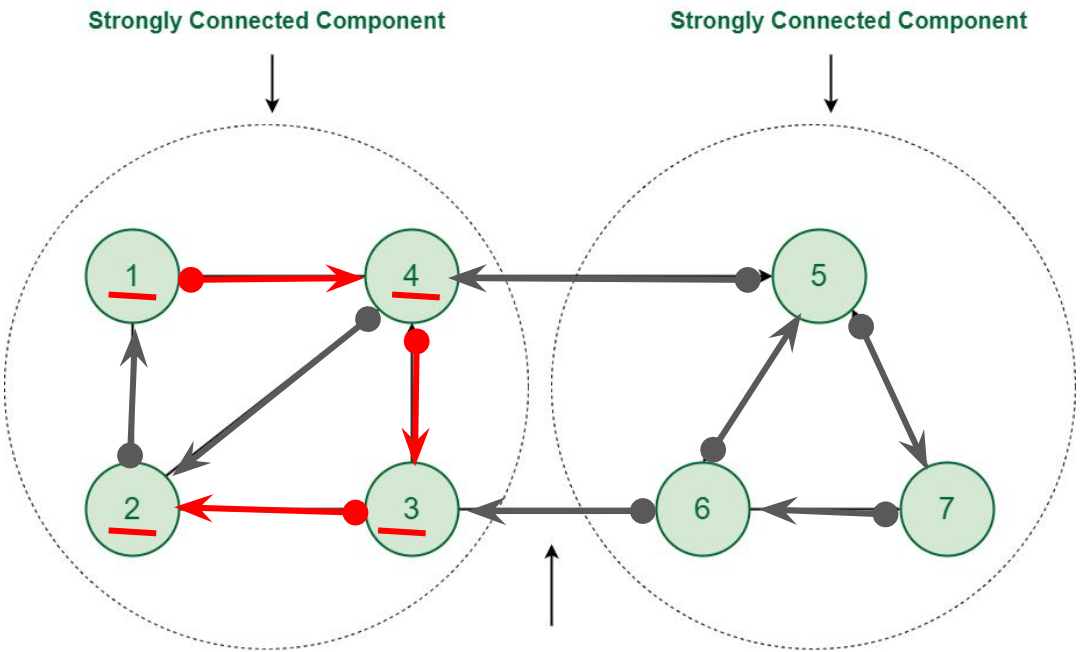
Dfs trace: 1,2,3,4,5,6,7



Phase 2: Find SCCs within the unidirectional components by reverse DFS

Intuition of Kosaraju's (Phase 2)

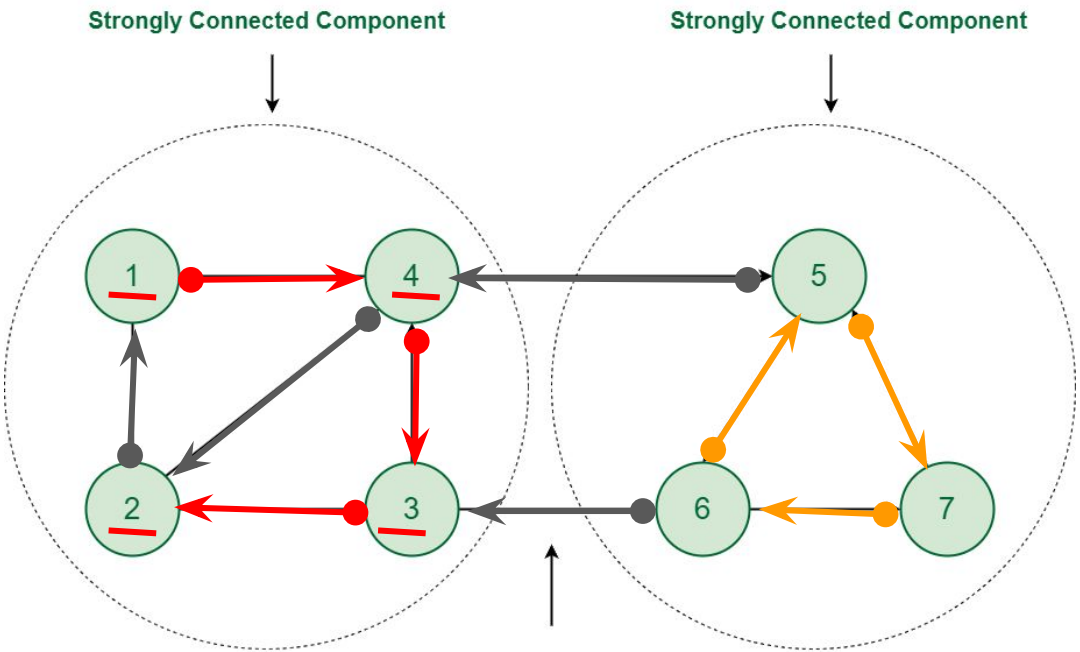
Dfs trace: 1,2,3,4,5,6,7



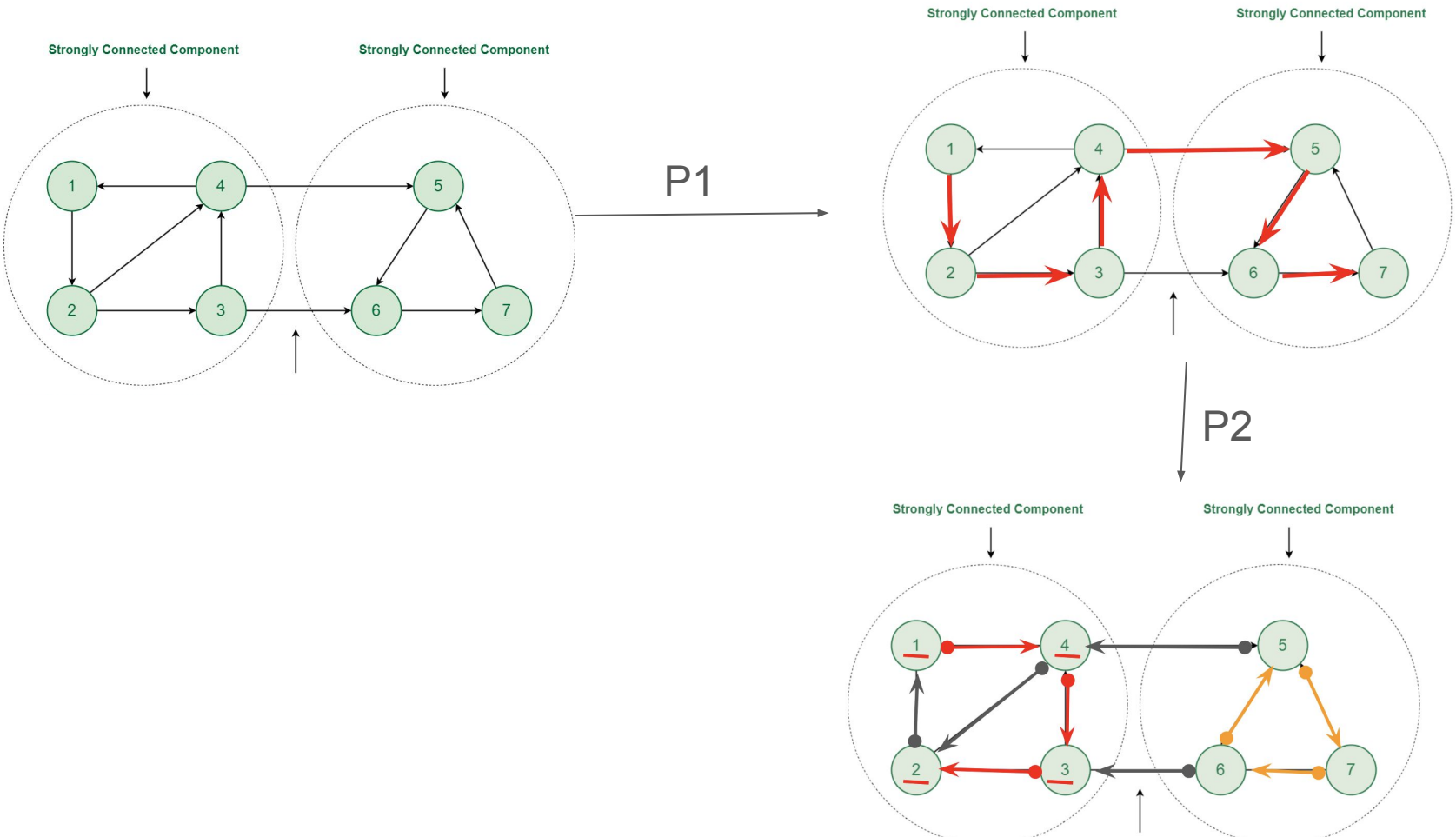
Phase 2: Find SCCs within the unidirectional components by reverse DFS

Intuition of Kosaraju's (Phase 2)

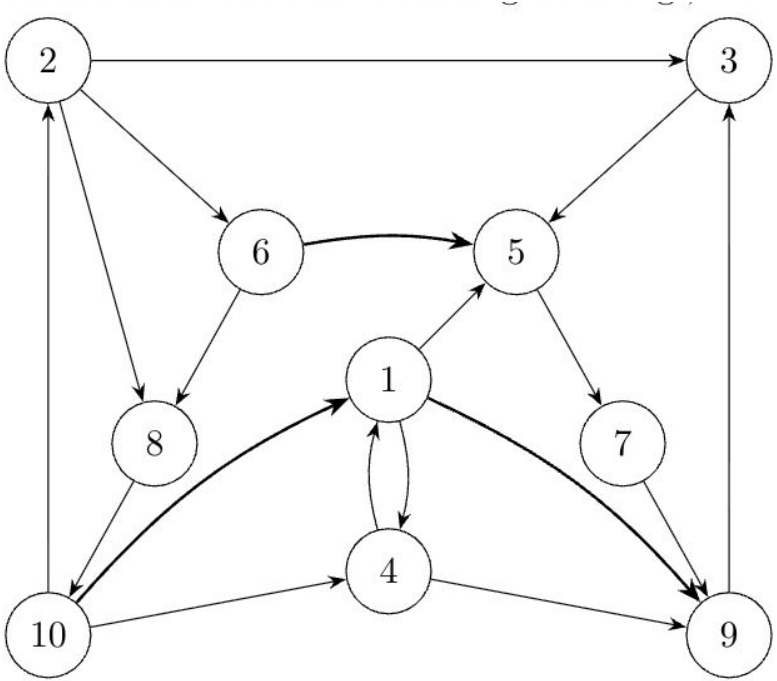
Dfs trace: 1,2,3,4,5,6,7



Phase 2: Find SCCs within the unidirectional components by reverse DFS


Strongly Connected Component
Strongly Connected Component
1
4
5
2
3
6
7
P1
Strongly Connected Component
Strongly Connected Component
1
4
5
2
3
6
7
P2
Strongly Connected Component
Strongly Connected Component
1
4
5
2
3
6
7

Phase 1: finding all connections



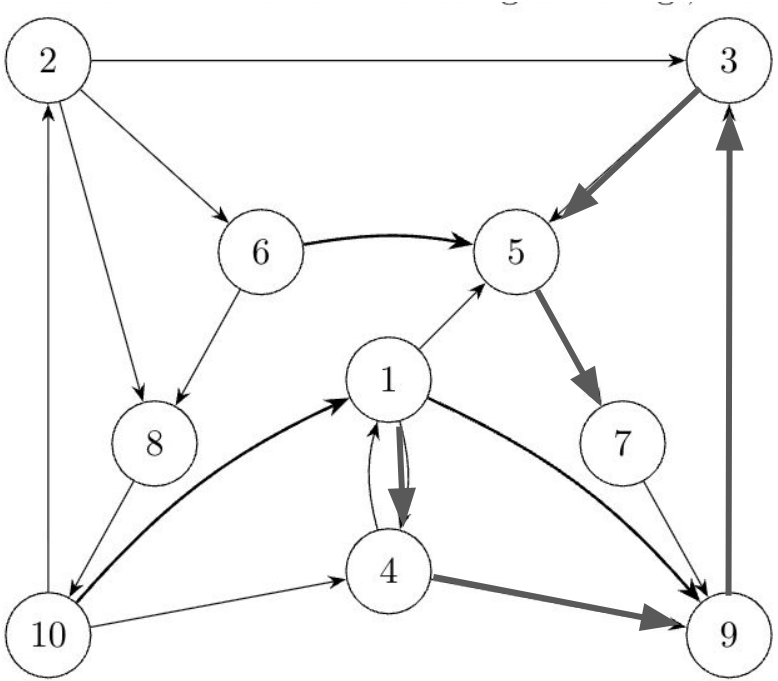
From v = 1,...,10:

Run DFS(v)

Transfer seen to L stack

//break once all nodes marked

Phase 1: finding all connections



From v = 1,...,10:

Run DFS(v)

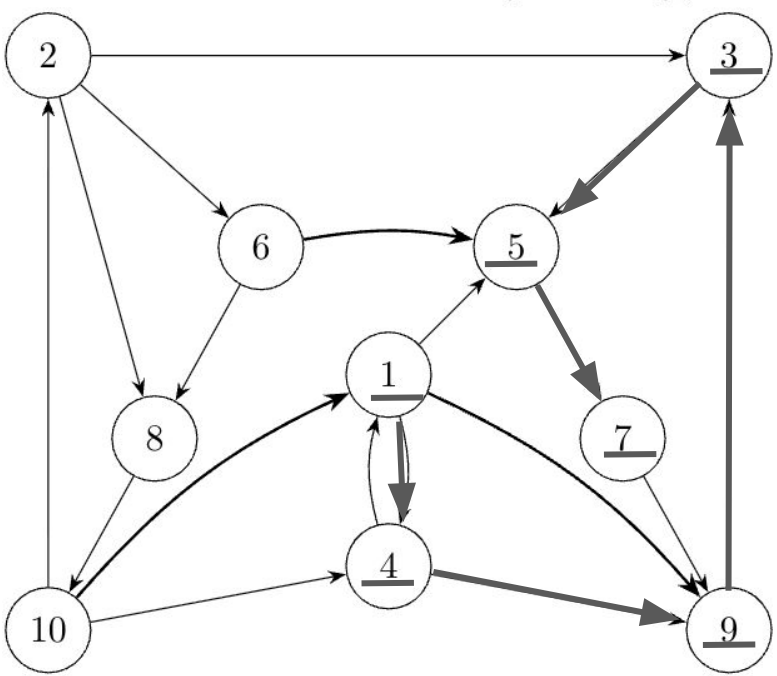
Transfer seen stack to L stack

//break once all nodes marked

7
5
3
9
4
1

Seen L

Phase 1: finding all connections



From v = 1,...,10:

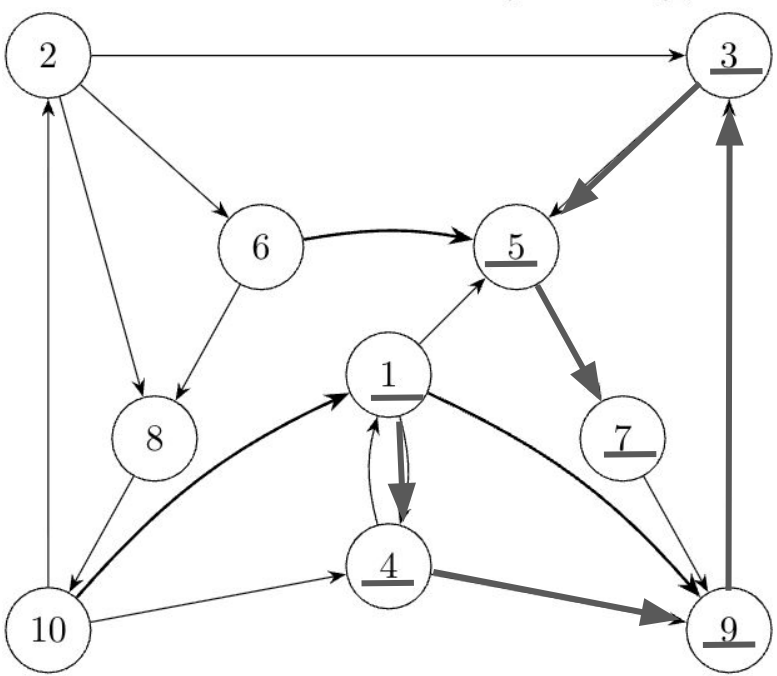
Run DFS(v)

Transfer seen stack to L stack

//break once all nodes marked

Seen	L
	1
	4
	9
	3
	5
	7

Phase 1: finding all connections



From v = 1,...,10:

Run DFS(v)

Skip to next unseen (2)

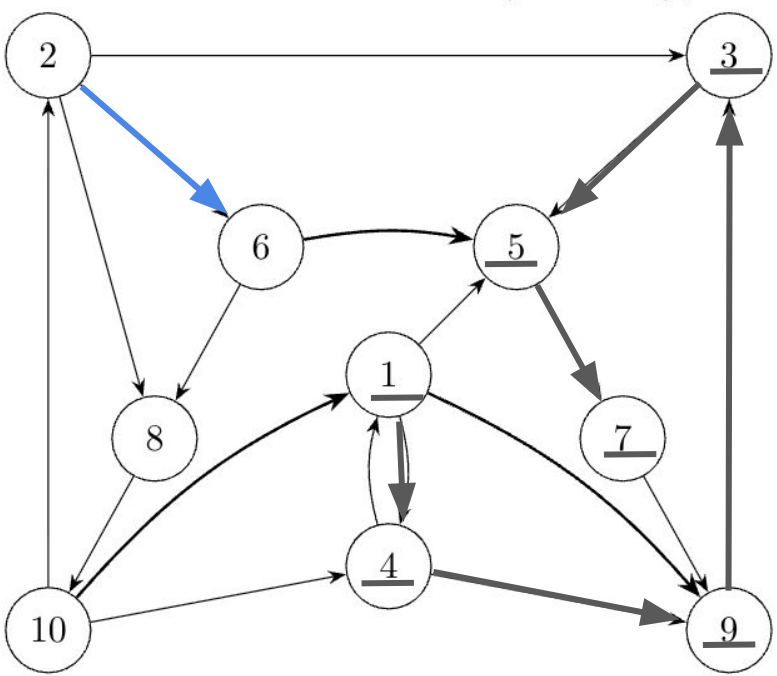
Transfer seen stack to L stack

//break once all nodes marked

1
4
9
3
5
7

Seen	L

Phase 1: finding all connections



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1
4
9
3
5
7

Seen	L

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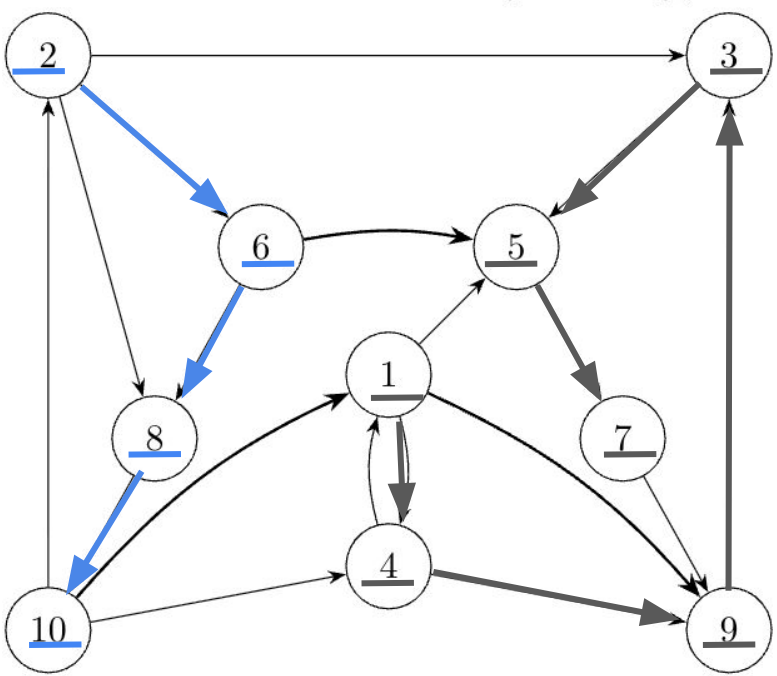

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Skip to next unseen (2)

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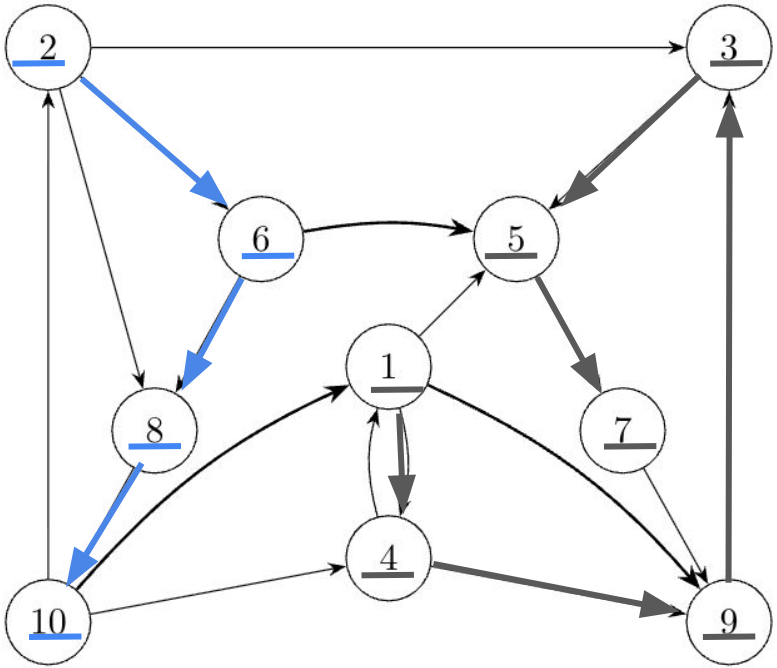
Seen	L
10	1
8	4
6	9
2	3
	5
	7

Phase 1: finding all connections



From v = 1,...,10:

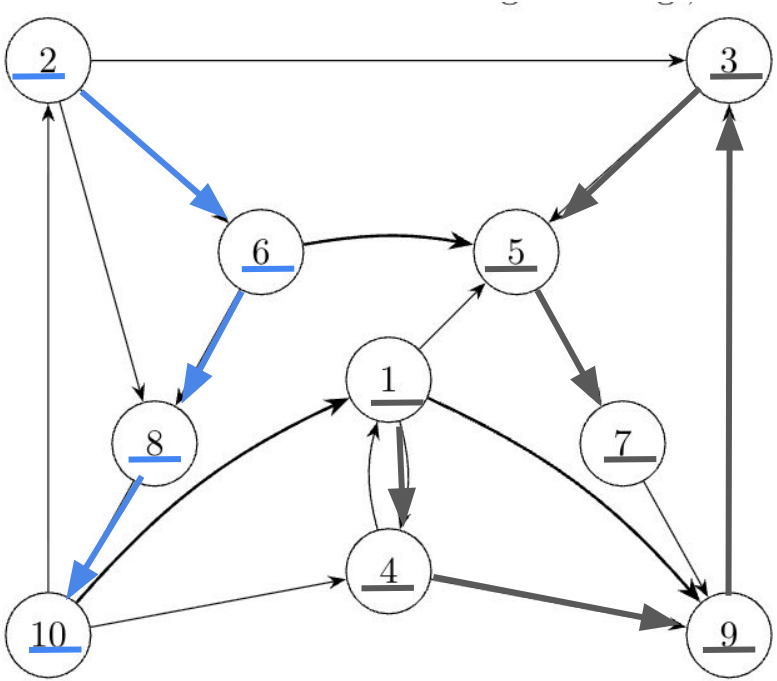
Run DFS(v)

Transfer seen stack to L stack

//break once all nodes marked

Seen	L
2	
6	
8	
10	
1	
4	
9	
3	
5	
7	

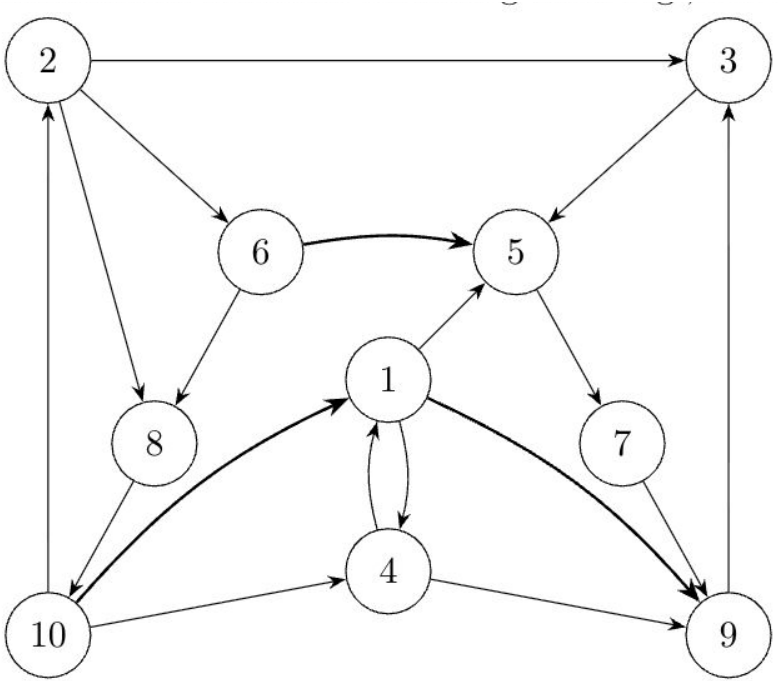
Phase 1: finding all connections



Insight from before:
SCCs are inside these DFS traversals

Seen	L
	2
	6
	8
	10
	1
	4
	9
	3
	5
	7

Phase 2: Reverse graph to filter out non-SCCs

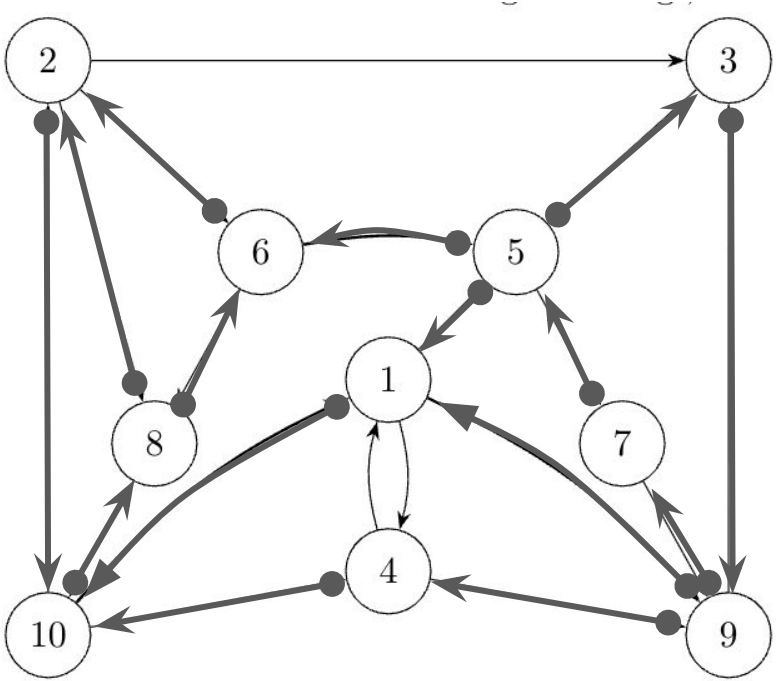


1. Reverse edges
2. While unmarked vertices:
   a. i = L.pop()
   b. //if i is marked, continue to next loop iter.
   c. dfs(i) //and mark as SCC j, j++

2
6
8
10
1
4
9
3
5
7

L

Phase 2: Reverse graph to filter out non-SCCs



PRO TIP: Bring a sharpie to the midterm

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2
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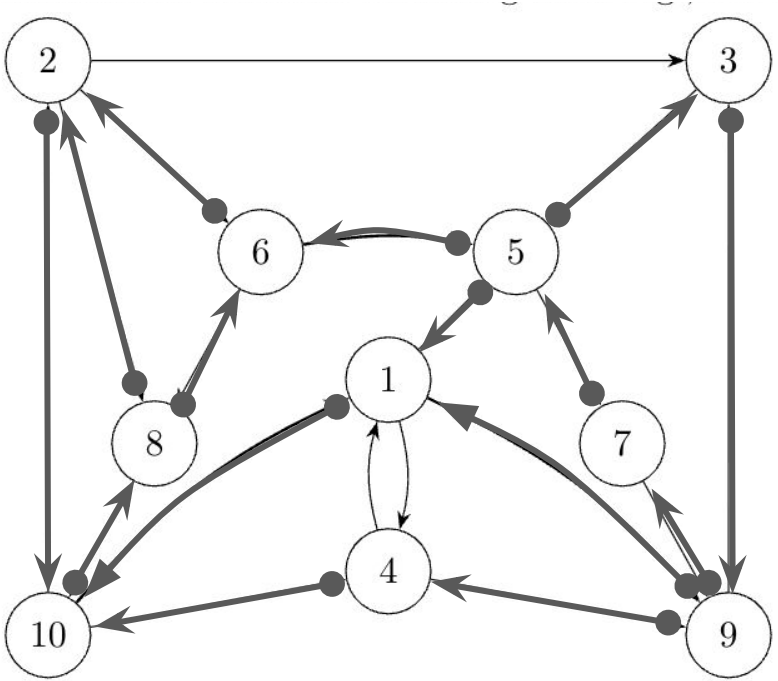
9
3
5
7
L

Phase 2: Reverse graph to filter out non-SCCs



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2
6
8
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1
4
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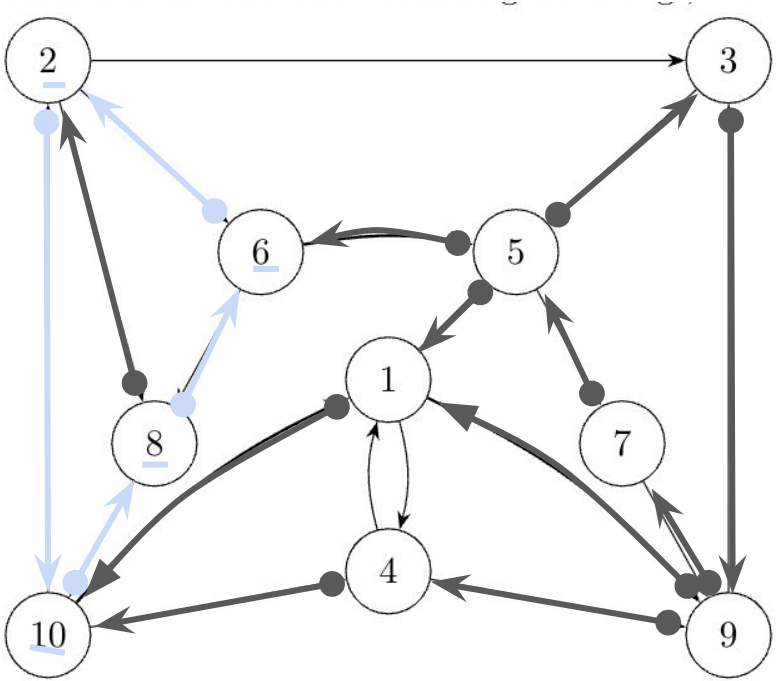
3
5
7
L

Phase 2: Reverse graph to filter out non-SCCs



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1
4
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5
7

L

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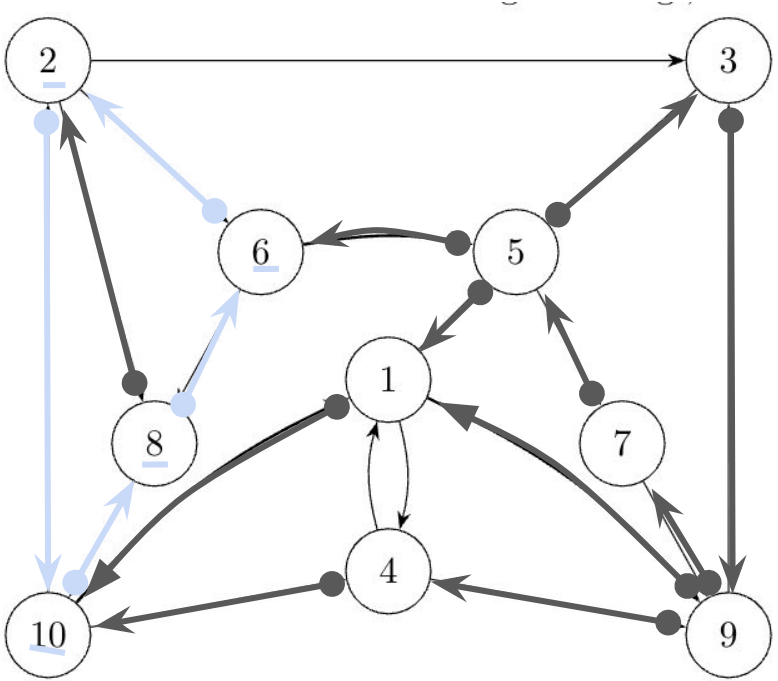

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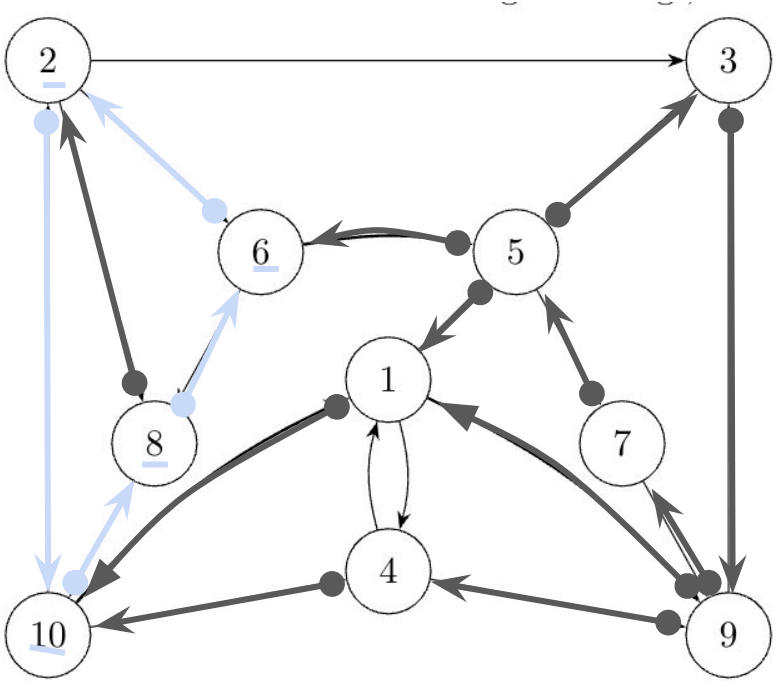
~~6~~
~~8~~
~~10~~
1
4
9
3
5
7

L

Phase 2: Reverse graph to filter out non-SCCs



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1
4
9
3
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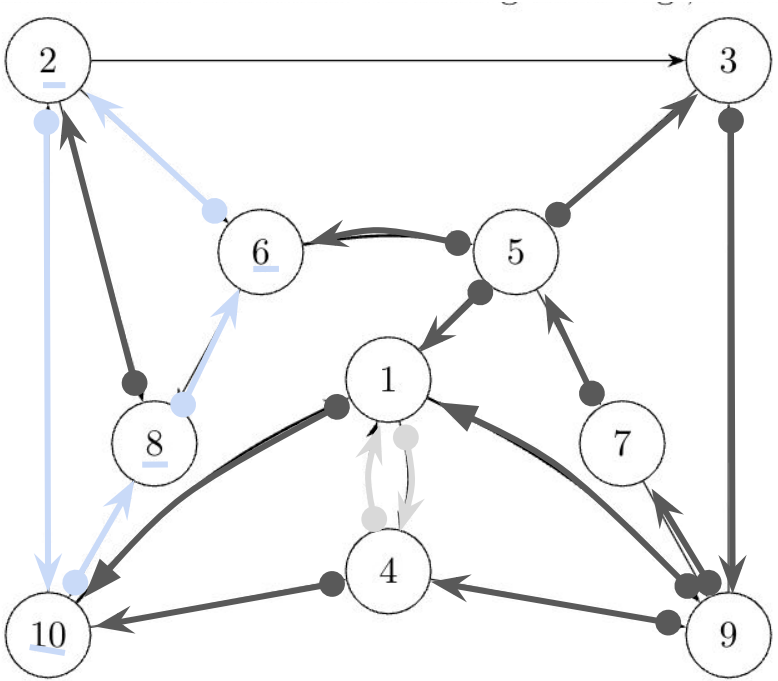
7

L

Phase 2: Reverse graph to filter out non-SCCs



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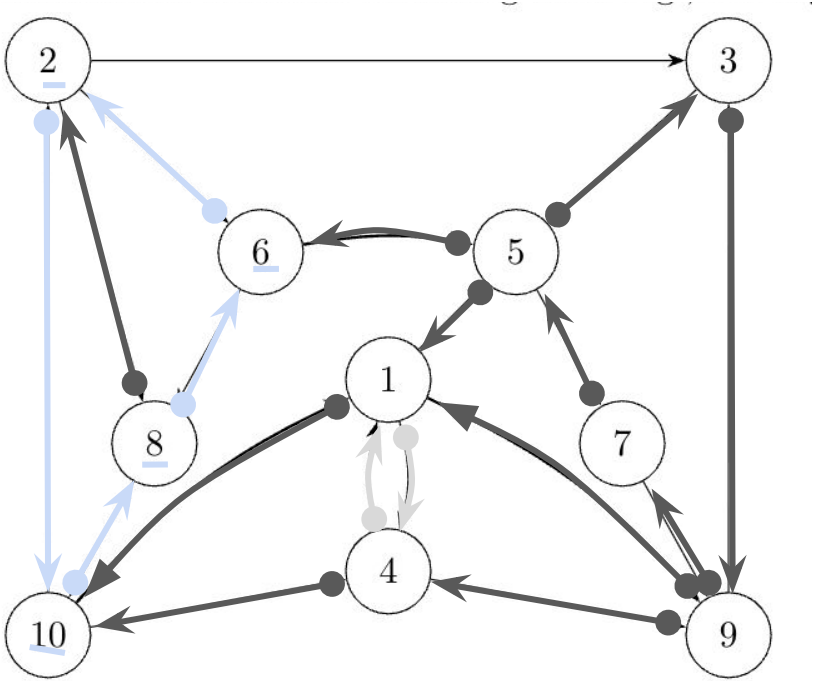
1
4
9
3
5
7

L

Phase 2: Reverse graph to filter out non-SCCs



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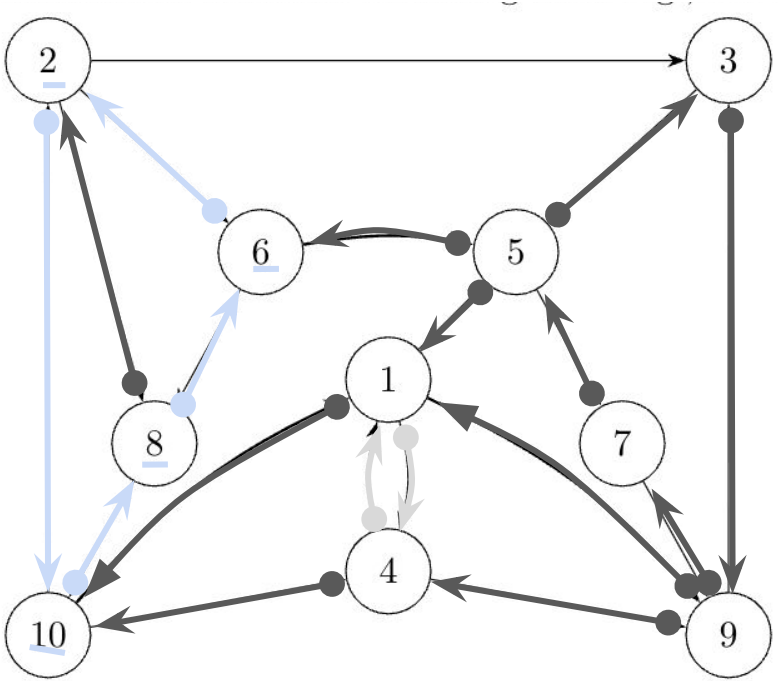
~~1~~
~~4~~
9
3
5
7

L

Phase 2: Reverse graph to filter out non-SCCs



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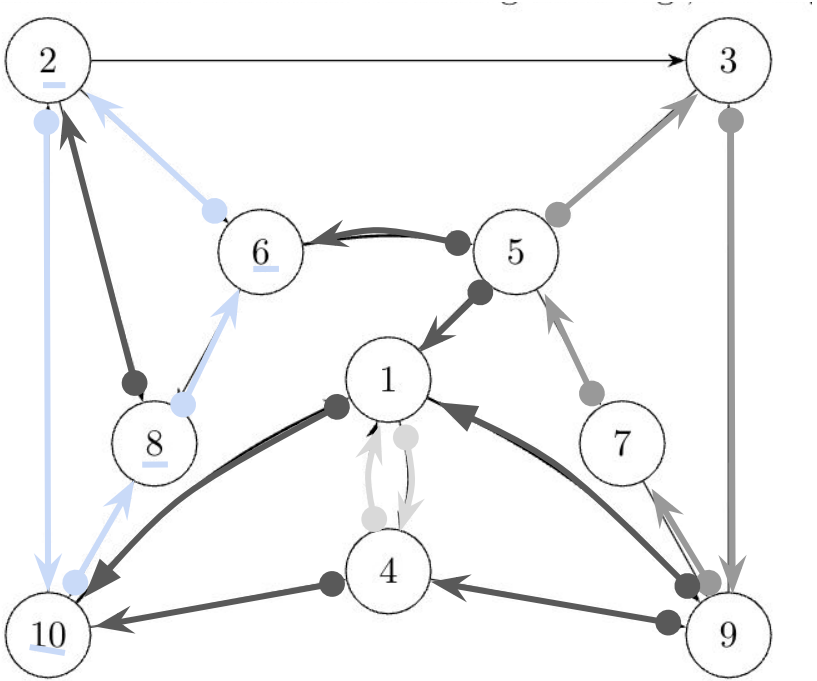
9
3
5
7
L

Phase 2: Reverse graph to filter out non-SCCs



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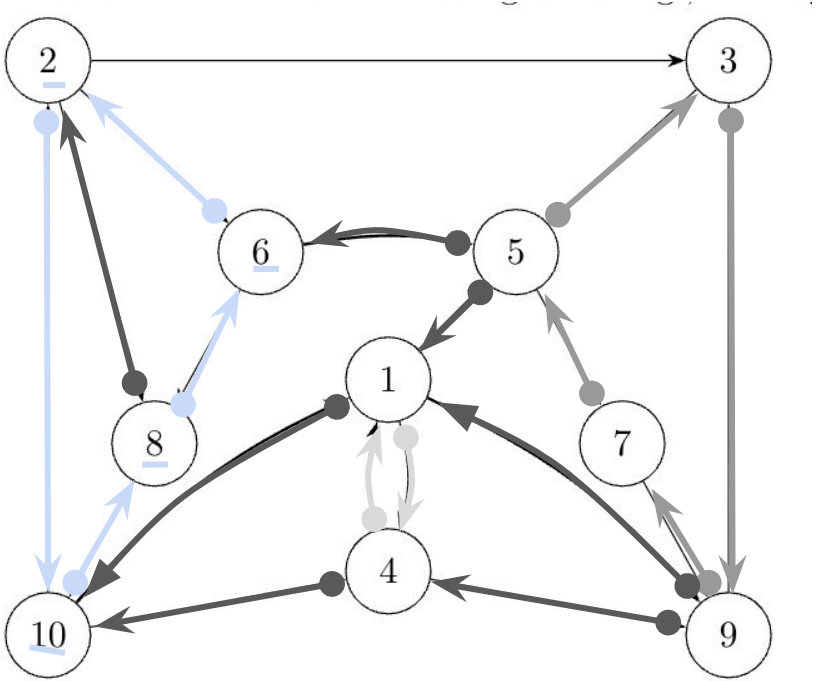
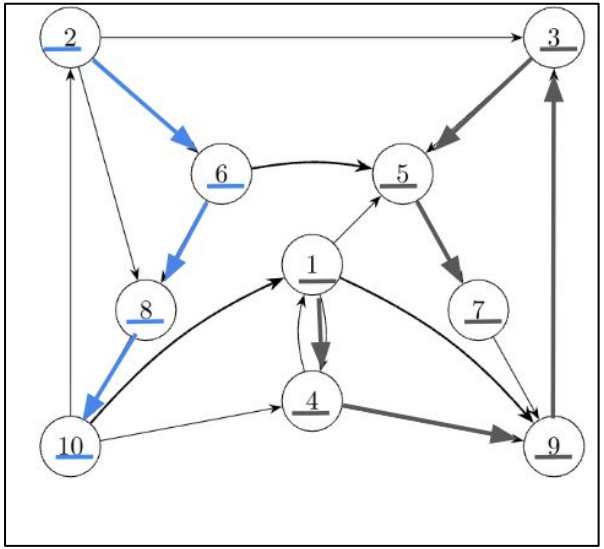
~~3~~
~~5~~
~~7~~
L

Summary: 3 SCCs



Recall the first phase



Partitioned the second (black) dfs traversal into 2 SCCs

Question 2

(Dijkstra's algorithm)

1. Give a simple example of a directed graph with negative-weighted edges for which Dijkstra's algorithm produces an incorrect answer.

2. Given a weighted, directed graph $G = (V, E)$ in which edges that leave the source vertex s may have negative weights, but all other edge weights are non-negative, and there are no negative-weighted cycles. Can the Dijkstra's algorithm correctly find all the shortest paths from s in this graph?

3. Your classmate claims that Dijkstra's algorithm relaxes the edges of every shortest path in the graph in the order in which they appear on the path. Show that him/her is mistaken by constructing a directed graph for which the Dijkstra's algorithm could relax the edges of a shortest path out of order.

Hint: The shortest path between two vertices in the graph is not necessarily unique.

Dijkstra - Find shortest paths between (input starting vertex s) and all other vertices.

- **Single source shortest paths**

Dijkstra

```
algorithm DijkstraShortestPath(G(V,E), s ∈ V)

   let dist:V → ℤ
   let prev:V → V
   let Q be an empty priority queue

   dist[s] ← 0
   for each v ∈ V do
      if v ≠ s then
         dist[v] ← ∞
      end if
      prev[v] ← -1
      Q.add(dist[v], v)
   end for

   while Q is not empty do
      u ← Q.getMin()
      for each w ∈ V adjacent to u still in Q do
         d ← dist[u] + weight(u, w)
         if d < dist[w] then
            dist[w] ← d
            prev[w] ← u
            Q.set(d, w)
         end if
      end for
   end while

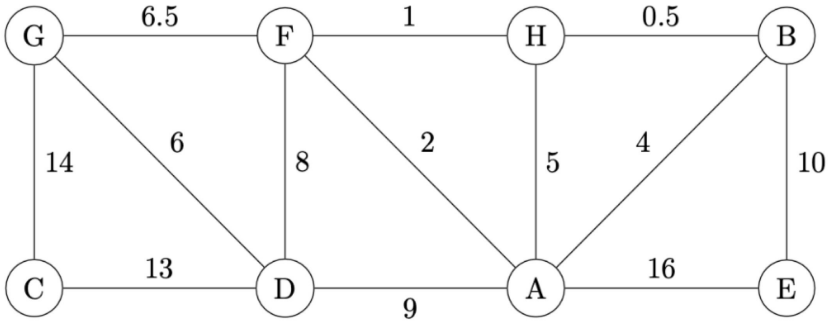
   return dist, prev
end algorithm
```

For a vertex s, finds shortest paths to all vertices. At each step..

- Consider current closest vertex u (priority queue)
- Greedily update path lengths to u's neighbors
  - **"Relaxing the edge"**
- Mark as visited

For your studies, a walkthrough of how Dijstra works..

Start at A

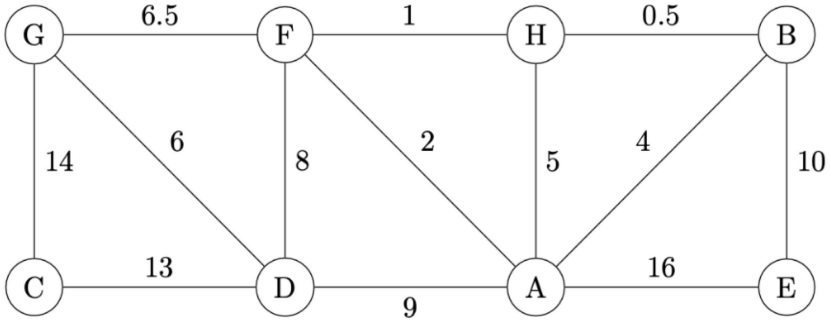


	A	B	C	D	E	F	G	H
dist	0	∞	∞	∞	∞	∞	∞	∞
prev	-1	-1	-1	-1	-1	-1	-1	-1

Q
(0, A)
(∞, B)
(∞, C)
(∞, D)
(∞, E)
(∞, F)
(∞, G)

```
for each w ∈ V adjacent to u still in Q do
    d ← dist[u] + weight(u, w)
    if d < dist[w] then
        dist[w] ← d
        prev[w] ← u
        Q.set(d, w)
```

Update based on A's edges



	A	B	C	D	E	F	G	H
dist	0	9	∞	9	16	2	∞	5
prev	-1	A	-1	A	A	A	-1	A

```
for each w ∈ V adjacent to u still in Q do
    d ← dist[u] + weight(u, w)
    if d < dist[w] then
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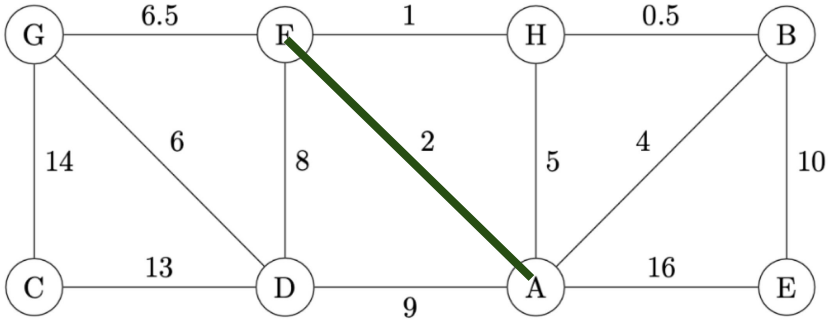
        Q.set(d, w)
```

Q

(2, F)
(4, B)
(9, D)
(16, E)
(∞, C)
(∞, G)

Next on the prior. Q is F



	A	B	C	D	E	F	G	H
dist	0	9	∞	9	16	2	∞	5
prev	-1	A	-1	A	A	A	-1	A

```
for each w ∈ V adjacent to u still in Q do
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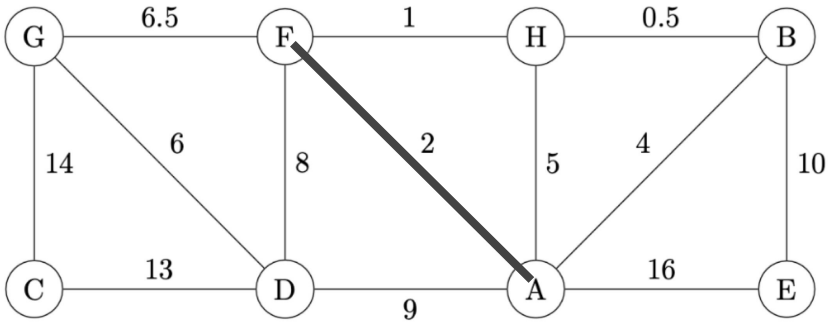
```

Q

(2, F)
(4, B)
(9, D)
(16, E)
(∞, C)
(∞, G)

Why didn't we update D?



	A	B	C	D	E	F	G	H
dist	0	4	∞	9	16	2	8.5	3
prev	-1	A	-1	A	A	A	F	F

```
for each w ∈ V adjacent to u still in Q do
    d ← dist[u] + weight(u, w)
    if d < dist[w] then
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        Q.set(d, w)
```

Q

(3, H)

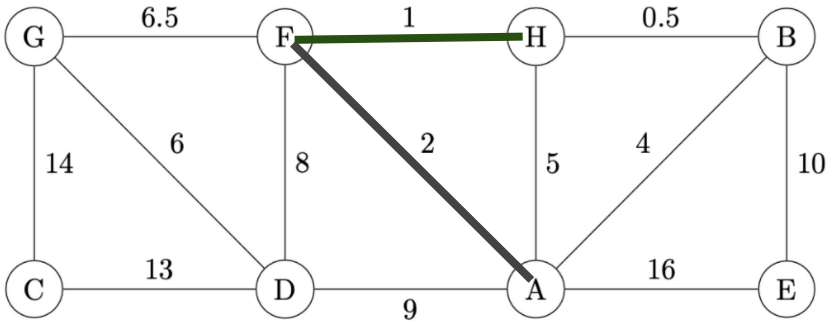
(4, B)

(9, D)

(16, E)

(∞, C)

(8.5, G)



	A	B	C	D	E	F	G	H
dist	0	9	∞	9	16	2	8.5	3
prev	-1	A	-1	A	A	A	F	F

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        dist[w] ← d
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        Q.set(d, w)
```

Q

(3, H)

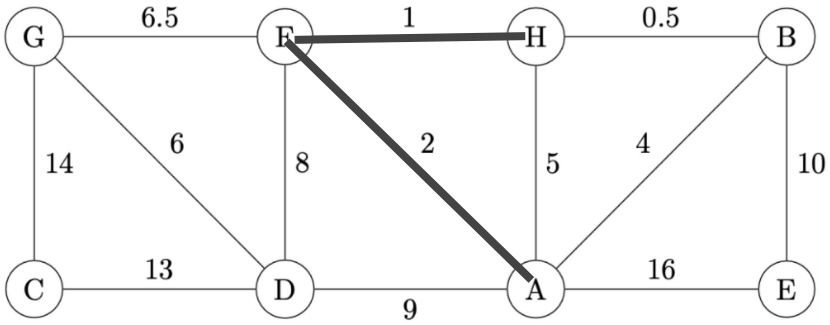
(4, B)

(9, D)

(16, E)

(∞, C)

(8.5, G)




```
for each w ∈ V adjacent to u still in Q do
    d ← dist[u] + weight(u, w)
    if d < dist[w] then
        dist[w] ← d
        prev[w] ← u
        Q.set(d, w)
```

	A	B	C	D	E	F	G	H
dist	0	3.5	∞	9	16	2	8.5	3
prev	-1	H	-1	A	A	A	F	F

Q

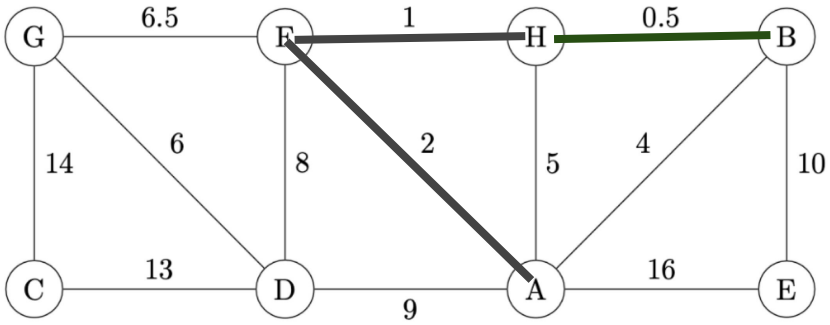
(3.5, B)

(8.5, G)

(9, D)

(16, E)

(∞, C)

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```

	A	B	C	D	E	F	G	H
dist	0	3.5	∞	9	16	2	8.5	3
prev	-1	H	-1	A	A	A	F	F

Q

(3.5, B)

(8.5, G)

(9, D)

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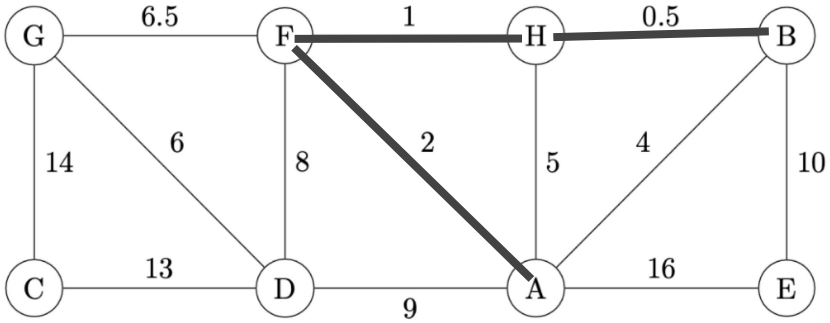
(∞, C)



	A	B	C	D	E	F	G	H
dist	0	3.5	∞	9	13.5	2	8.5	3
Prev	-1	H	-1	A	B	A	F	F

```
for each w ∈ V adjacent to u still in Q do
    d ← dist[u] + weight(u, w)
    if d < dist[w] then
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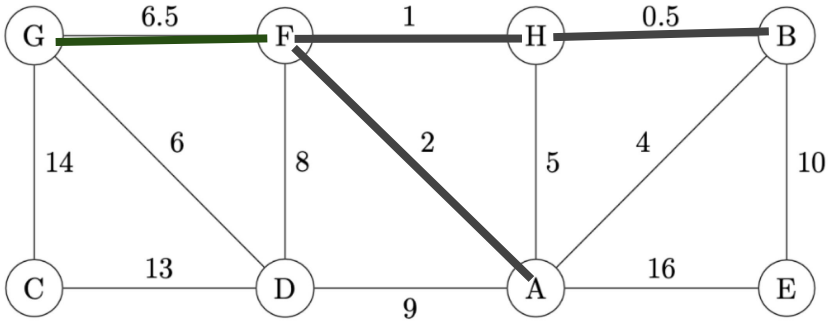
Q

(8.5, G)
(9, D)
(13.5, E)
(∞, C)



	A	B	C	D	E	F	G	H
dist	0	3.5	∞	9	13.5	2	8.5	3
Prev	-1	H	-1	A	B	A	F	F

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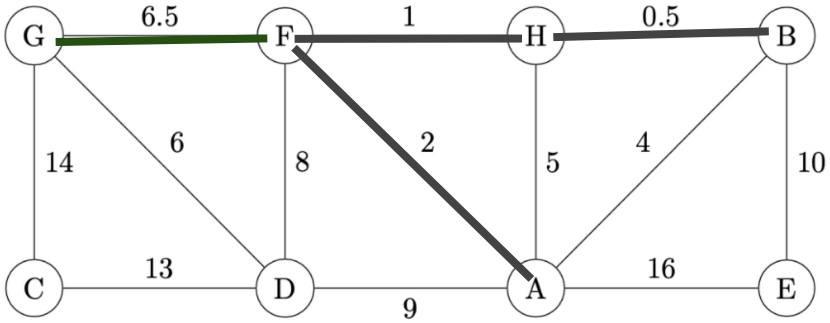
```

Q

(8.5, G)
(9, D)
(13.5, E)
(∞, C)



	A	B	C	D	E	F	G	H
dist	0	3.5	22.5	9	13.5	2	8.5	3
Prev	⊥	H	G	A	B	A	F	F

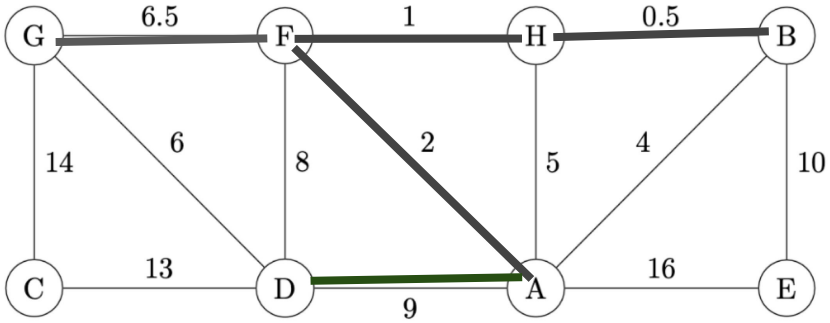
```
for each w ∈ V adjacent to u still in Q do
    d ← dist[u] + weight(u, w)
    if d < dist[w] then
        dist[w] ← d
        prev[w] ← u
        Q.set(d, w)
```

Q

(9, D)

(13.5, E)

(22.5, C)



	A	B	C	D	E	F	G	H
dist	0	3.5	22.5	9	13.5	2	8.5	3
Prev	-1	H	G	A	B	A	F	F

```
for each w ∈ V adjacent to u still in Q do
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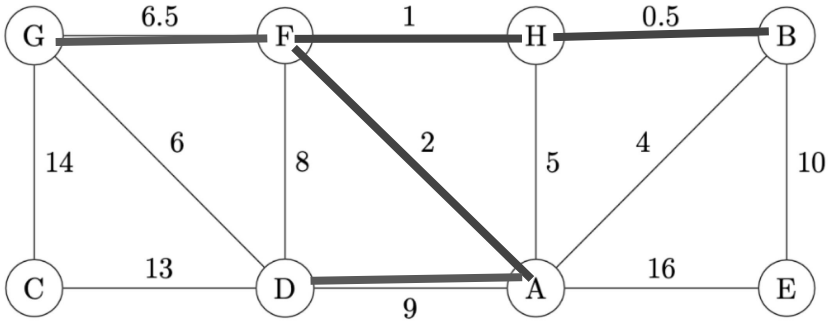
Q

(9, D)

(13.5, E)

(22.5, C)



	A	B	C	D	E	F	G	H
dist	0	3.5	22	9	13.5	2	8.5	3
Prev	⊥	H	D	A	B	A	F	F

Q

(13.5, E)

(22, C)

```
for each w ∈ V adjacent to u still in Q do
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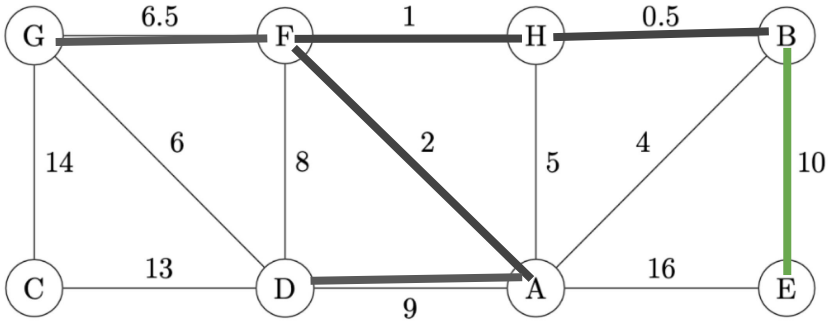
```



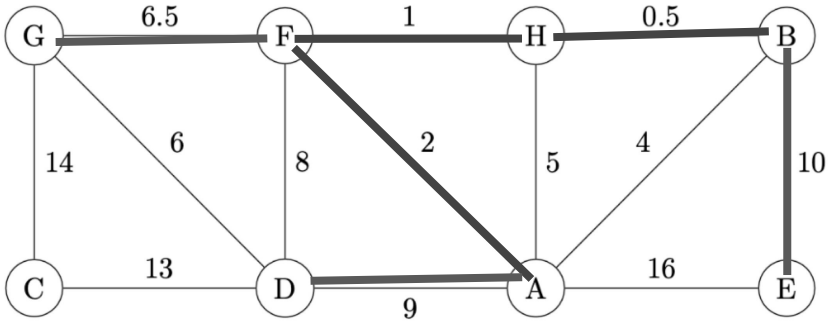
	A	B	C	D	E	F	G	H
dist	0	3.5	22	9	13.5	2	8.5	3
Prev	⊥	H	D	A	B	A	F	F

Q

(13.5, E)

(22 , C)

```
for each w ∈ V adjacent to u still in Q do
    d ← dist[u] + weight(u, w)
    if d < dist[w] then
        dist[w] ← d
        prev[w] ← u
        Q.set(d, w)
```

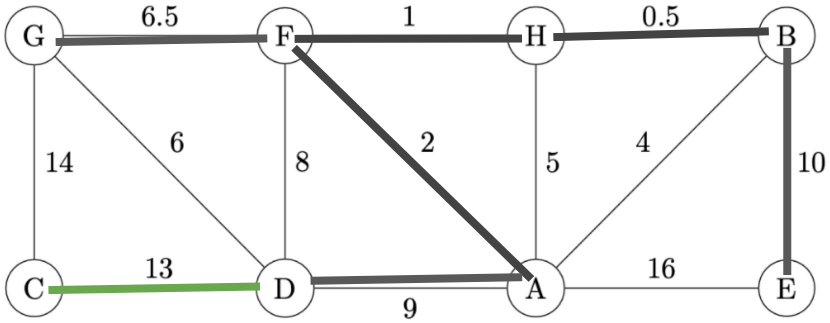


	A	B	C	D	E	F	G	H
dist	0	3.5	22	9	13.5	2	8.5	3
Prev	-1	H	D	A	B	A	F	F

```
for each w ∈ V adjacent to u still in Q do
    d ← dist[u] + weight(u, w)
    if d < dist[w] then
        dist[w] ← d
        prev[w] ← u
        Q.set(d, w)
```

Q

(22 , C)



	A	B	C	D	E	F	G	H
dist	0	3.5	22	9	13.5	2	8.5	3
Prev	-1	H	D	A	B	A	F	F

```
for each w ∈ V adjacent to u still in Q do
    d ← dist[u] + weight(u, w)
    if d < dist[w] then
        dist[w] ← d
        prev[w] ← u
        Q.set(d, w)
```

Q

(22 , C)

Question 2

(Dijkstra's algorithm)

1. Give a simple example of a directed graph with negative-weighted edges for which Dijkstra's algorithm produces an incorrect answer.

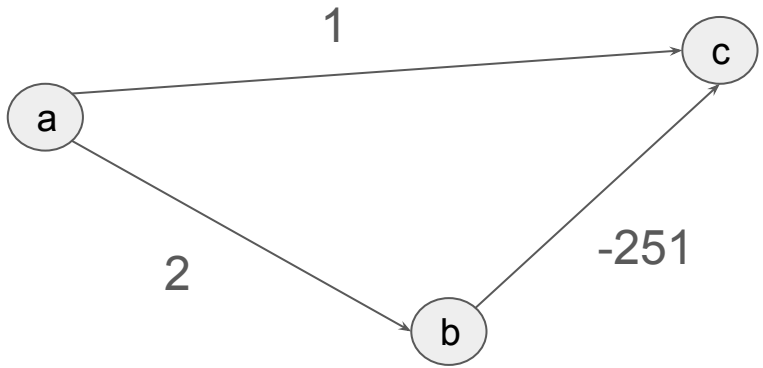
Question 1

(Dijkstra's algorithm)

1. Give a simple example of a directed graph with negative-weighted edges for which Dijkstra's algorithm produces an incorrect answer.

Intuition: once a vertex is removed from the pQ, its shortest path is fixed.

negative edge *may* be at the end, Dijkstra won't encounter it in time



pQ

1. (a,0)
2. (b,inf)
3. (c,inf)

	A	B	C
dist	0	inf	inf
prev	A	-1	-1

Question 1

(Dijkstra's algorithm)

1. Give a simple example of a directed graph with negative-weighted edges for which Dijkstra's algorithm produces an incorrect answer.

Intuition: once a vertex is removed from the pQ, its shortest path is fixed.

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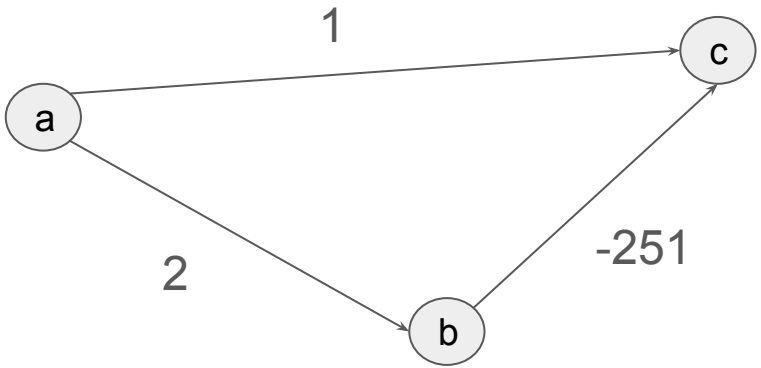

pQ

1. (c,**1**)
2. (b,**2)**

	A	B	C
dist	0	2	1
prev	A	-1	-1

Question 1

(Dijkstra's algorithm)

1. Give a simple example of a directed graph with negative-weighted edges for which Dijkstra's algorithm produces an incorrect answer.

Intuition: once a vertex is removed from the pQ, its shortest path is fixed.

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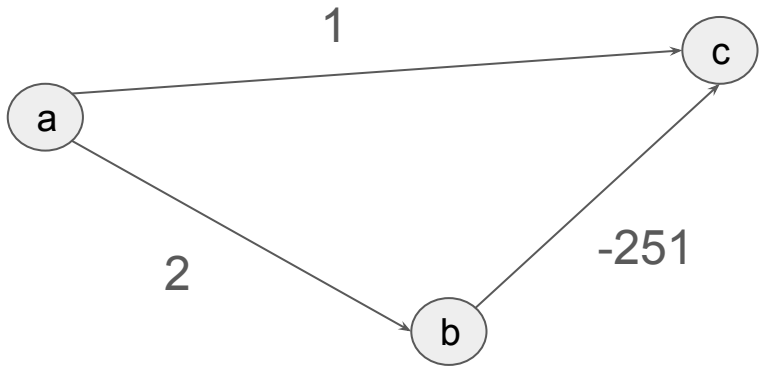


pQ

1. ~~(c,**1**)~~
2. (b,**2**)

	A	B	C
dist	0	2	1
prev	A	-1	-1

We take c off here, but there is clearly a shorter (negative) path!

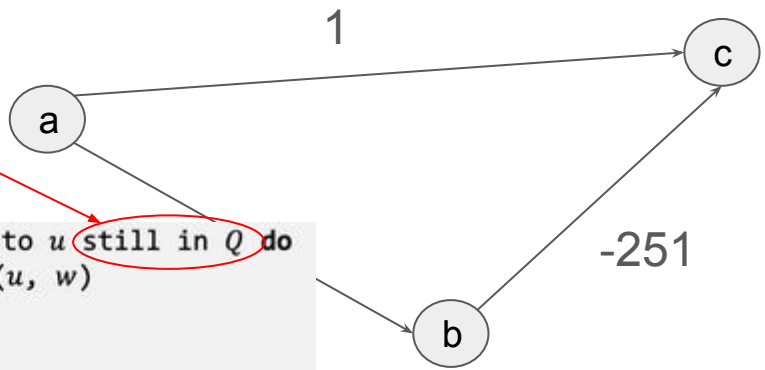
Question 1

(Dijkstra's algorithm)

1. Give a simple example of a directed graph with negative-weighted edges for which Dijkstra's algorithm produces an incorrect answer.

Intuition: once a vertex is removed from the pQ, its shortest path is fixed.

negative edge *may* be at the end, Dijkstra won't encounter it in time




```
or each w ∈ V adjacent to u still in Q do
  d ← dist[u] + weight(u, w)
  if d < dist[w] then
     dist[w] ← d
     prev[w] ← u
     Q.set(d, w)
```

pQ

1. (b,**2**)

		B	C
dist	0	2	1
prev	A	-1	-1

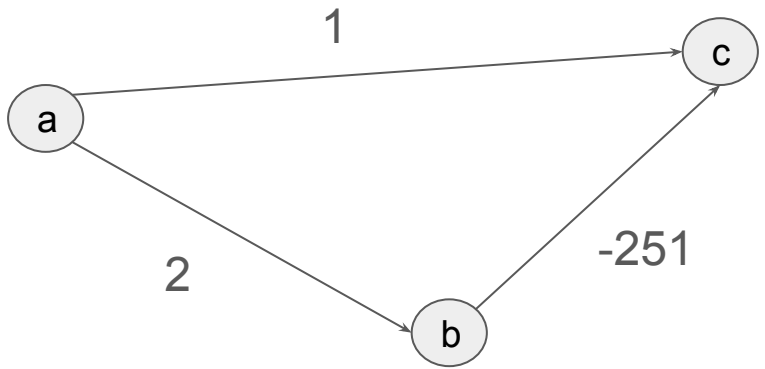
By the time we encounter b and find the path to c, c no longer in the pQ!!!

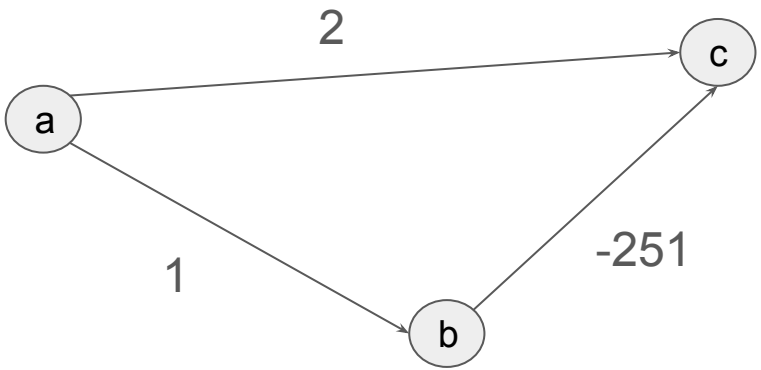
2. Given a weighted, directed graph $G = (V, E)$ in which edges that leave the source vertex s may have negative weights, but all other edge weights are non-negative, and there are no negative-weighted cycles. Can the Dijkstra's algorithm correctly find all the shortest paths from s in this graph?

This seems plausible…

Can we update the previous example so

that it actually finds the shortest path a -> c



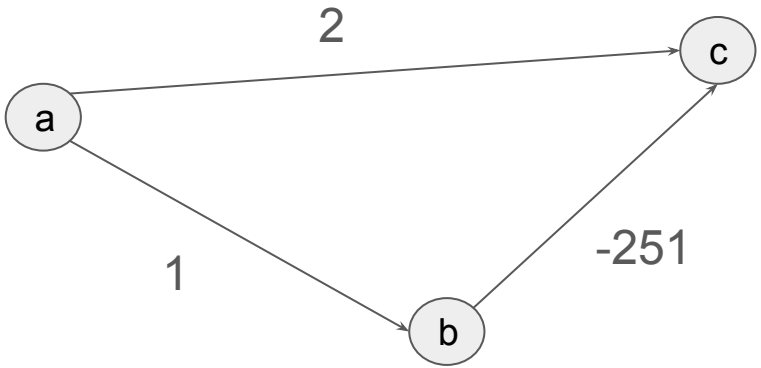


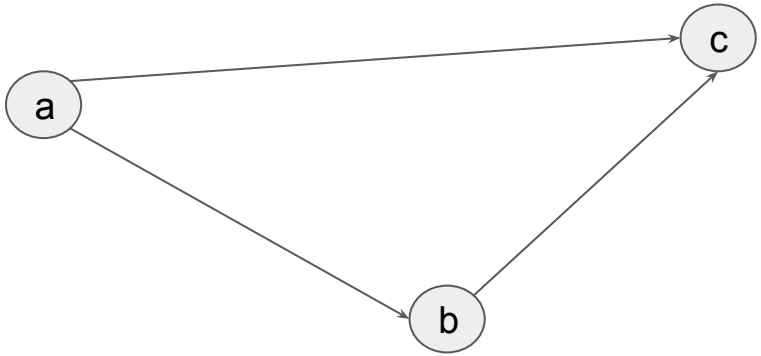
pQ

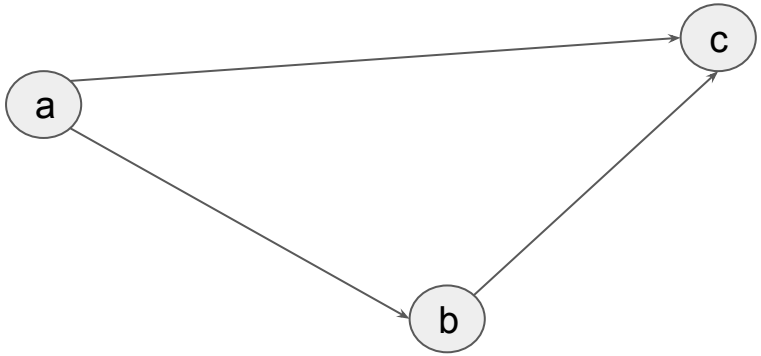
1.
2.
3.

	A	B	C
dist			
prev			

Other examples that work







2. Given a weighted, directed graph $G = (V, E)$ in which edges that leave the source vertex s may have negative weights, but all other edge weights are non-negative, and there are no negative-weighted cycles. Can the Dijkstra's algorithm correctly find all the shortest paths from s in this graph?

Another way to see this

Update step correct regardless if negative edge

We only have an error **if we don't update (see prev example)**

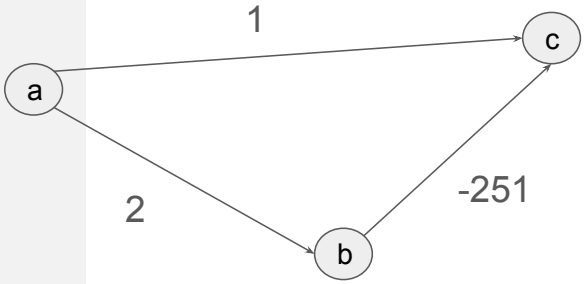
```
algorithm DijkstraShortestPath(G(V,E), s ∈ V)

   let dist:V → ℤ
   let prev:V → V
   let Q be an empty priority queue

   dist[s] ← 0
   for each v ∈ V do
      if v ≠ s then
         dist[v] ← ∞
      end if
      prev[v] ← -1
      Q.add(dist[v], v)
   end for

   while Q is not empty do
      u ← Q.getMin()
      for each w ∈ V adjacent to u still in Q do
         d ← dist[u] + weight(u, w)
         if d < dist[w] then
            dist[w] ← d
            prev[w] ← u
            Q.set(d, w)
         end if
      end for
   end while

   return dist, prev
end algorithm
```



3. Your classmate claims that Dijkstra's algorithm relaxes the edges of every shortest path in the graph in the order in which they appear on the path. Show that him/her is mistaken by constructing a directed graph for which the Dijkstra's algorithm could relax the edges of a shortest path out of order.

Hint: The shortest path between two vertices in the graph is not necessarily unique.

Relaxing an edge: checking if the current best known distance is better than the distance of the current edge

Basically, this step

```
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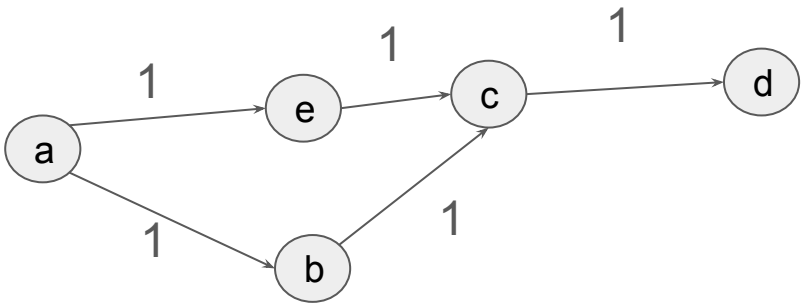
   while Q is not empty do
      u ← Q.getMin()
      for each w ∈ V adjacent to u still in Q do
         d ← dist[u] + weight(u, w)
         if d < dist[w] then
            dist[w] ← d
            prev[w] ← u
            Q.set(d, w)
         end if
      end for
   end while

   return dist, prev
end algorithm
```

Hint: The shortest path between two vertices in the graph is not necessarily unique.



	A	B	C	D	E
dist					
prev					

pQ

1.
2.
3.
4.
5.

Question 2

(Bellman-Ford algorithm)

For the Bellman-Ford algorithm, explain

1. why it only requires $|V| - 1$ passes?
2. why the last pass ($|V| - 1$) through the edges will determine if there are negative weight cycles or not?