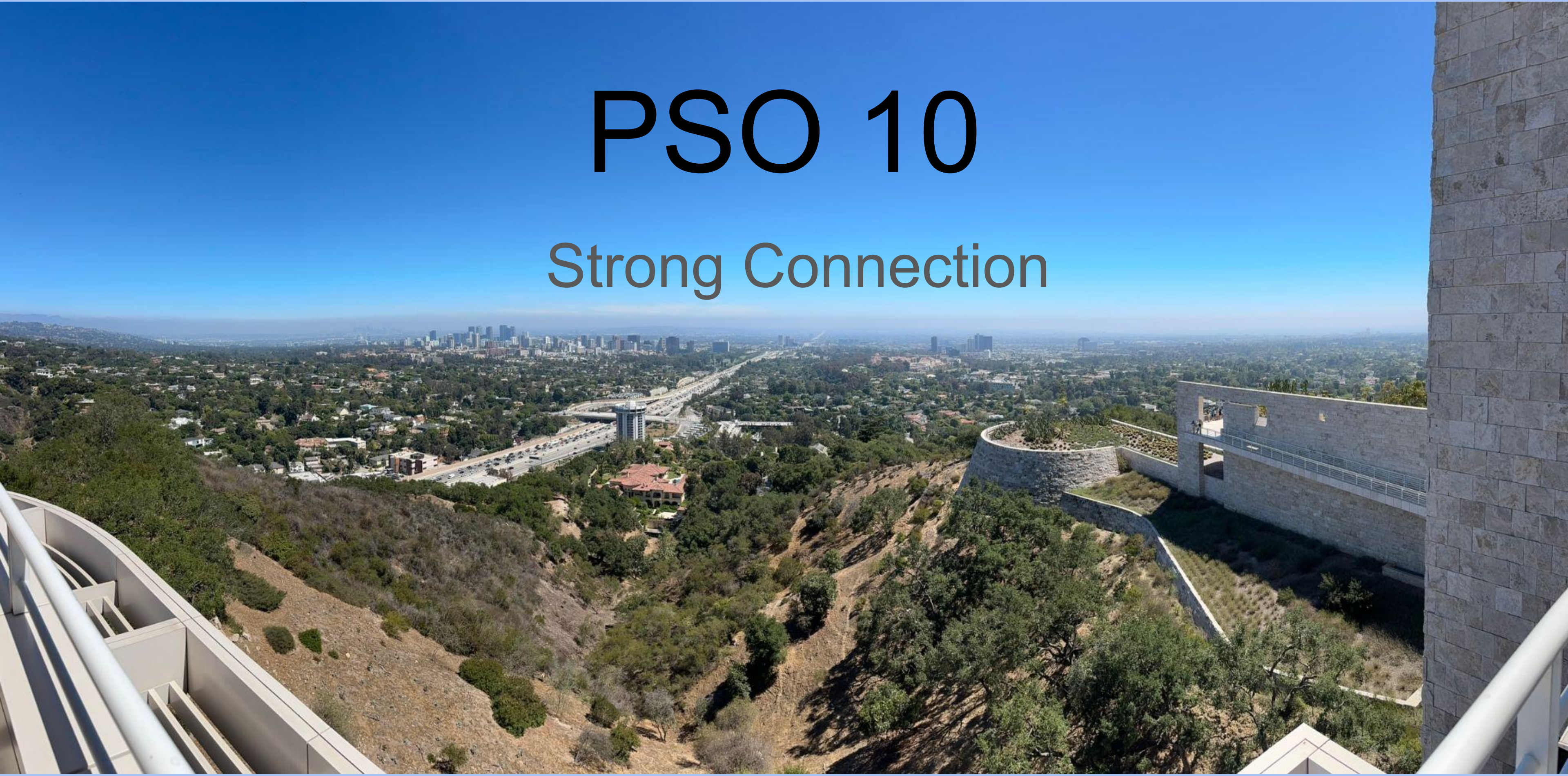
PSO 10

Strong Connection



https://justin-zhang.com/teaching/CS251

Hw5 - due Friday

- explicitly run deleteMin()

  donot transform into a 2-3 tree.

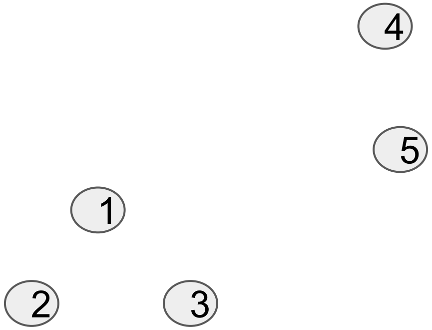
Question 1

(Strongly connected components)

1. How can the number of strongly connected components of a graph change if a new edge is added?

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

$$\text{in-degree}(v) = \text{out-degree}(v), \forall v \in V.$$



Question 2

Consider the directed graph $G = (V, E)$ given below:

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left] & B \arrow[r] \arrow[d] & C \arrow[r] & E \\
& D \arrow[urr] & &
\end{tikzcd}$$

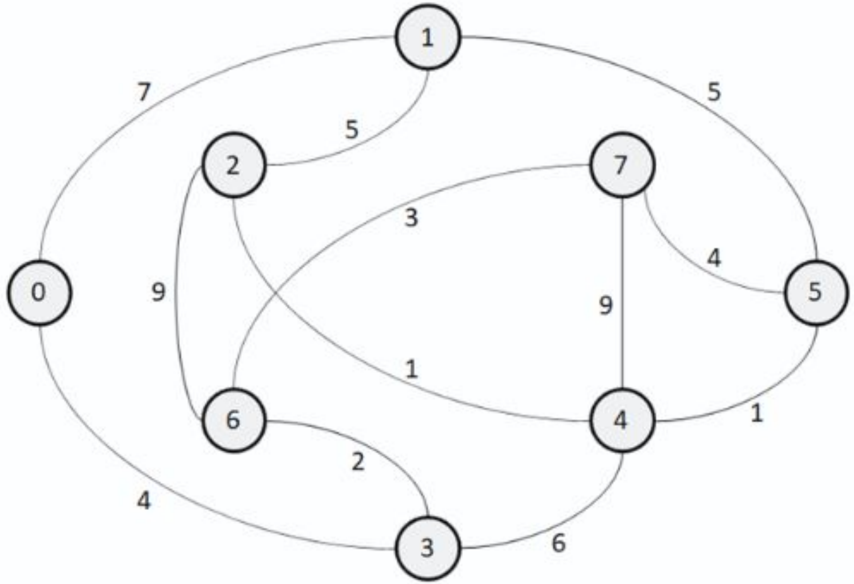
where the set of vertices is $V = \{A, B, C, D, E\}$ and the set of edges is:

$$E = \{(A, B), (A, C), (B, C), (B, D), (C, E), (D, E)\}.$$

1. Construct the adjacency matrix A of G.
2. Compute the transitive closure of G using Warshall's algorithm.
3. Draw the graph representation of the transitive closure of G.
4. Determine the reachability of each node in G.
5. Identify if G is strongly connected. If not, can you add one edge to make G become a strongly connected graph?

Question 3

Consider the following graph G:



Let G_d be a directed graph using the vertices of G. For a pair of vertices u and v connected by an edge in G, their respective directed edge in G_d is as follows:

$$\text{Edge with vertices } u \text{ and } v = \begin{cases} (u, v), & \deg(u) < \deg(v) \vee (\deg(u) = \deg(v) \wedge u < v) \\ (v, u), & \text{Otherwise} \end{cases}$$

1. Is G_d strongly connected? If yes, explain why. Otherwise, list the minimum number of edges required to make G_d strongly connected.
2. Show all the topological orderings of G_d.

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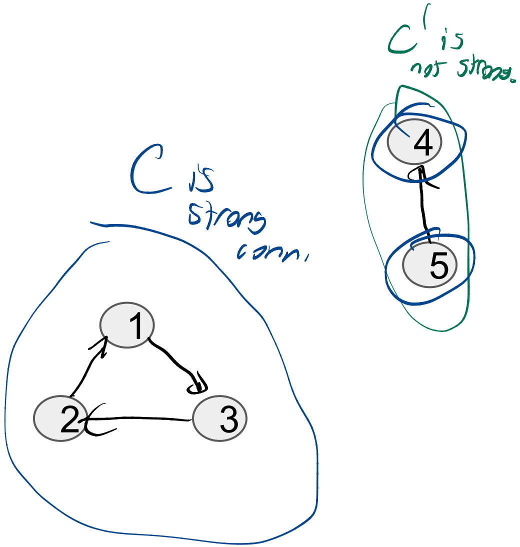
Strongly connected component?

$G = (V, E)$ (directed)

$C \subseteq V$, C is strongly connected component

if $\forall u, v \in C$: there is a path $u \rightsquigarrow v$.

C has maximal size.



Question 1

(Strongly connected components)

1. How can the number of strongly connected components of a graph change if a new edge is added?

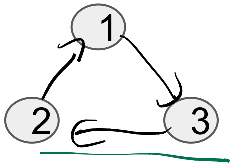
Can either increase/decrease/stay the same.

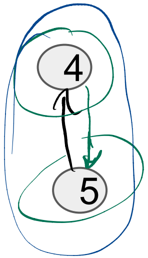
Can it **increase?**

add an edge

1. connected two strongly components

result: one strongly connected





Question 1

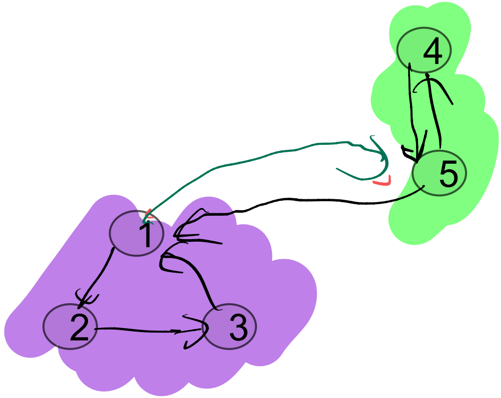
(Strongly connected components)

1. How can the number of strongly connected components of a graph change if a new edge is added?

Can either increase/decrease/stay the same.

Can it **increase? No**

Can it **decrease?** yes



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Can it **increase? No**

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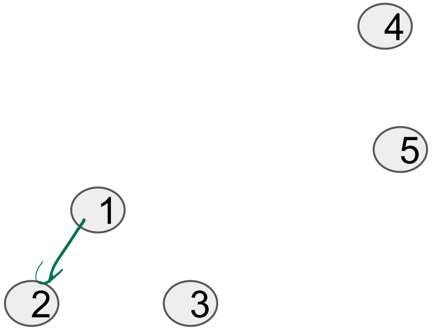
Can it **stay the same**?



5scc → 5scc

Question 1

(Strongly connected components)

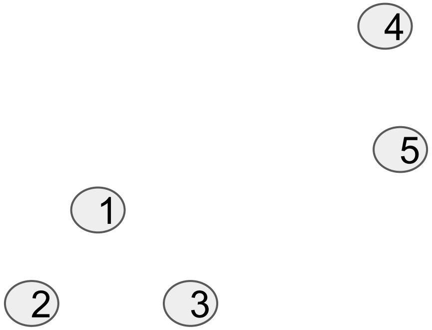
1. How can the number of strongly connected components of a graph change if a new edge is added?

Can either increase/decrease/stay the same.

Can it **increase? No**

Can it **decrease? Yes**

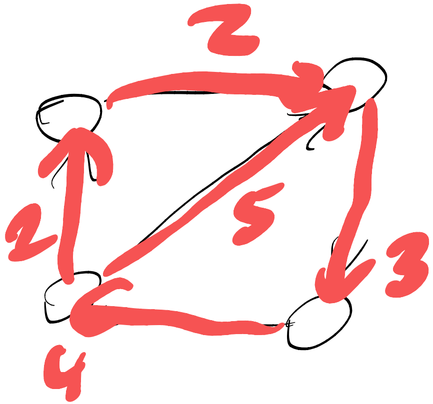
Can it **stay the same**? **Yes**



2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

$$\text{in-degree}(v) = \text{out-degree}(v), \forall v \in V.$$

Oh boy, lets start with an informal proof



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(→) Suppose G has an Euler tour.

We want to show every vertex v has indeg(v) = outdeg(v).

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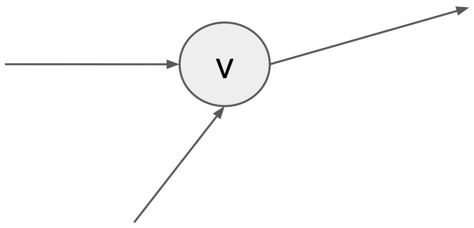
(→) Suppose G has an Euler tour.

We want to show every vertex v has indeg(v) = outdeg(v).



Suppose not, that there is a vertex v with indeg(v) > outdeg(v).

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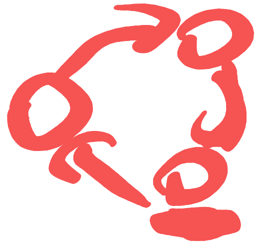

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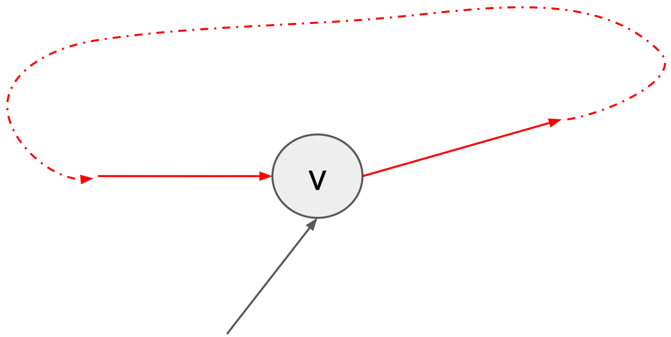

Suppose not, that there is a vertex v with indeg(v) > outdeg(v).

An Euler tour is a cycle i.e. each incoming edge is “paired” with an outgoing edge

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

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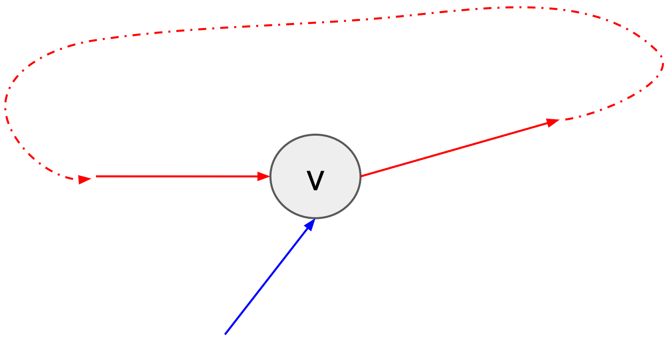

Suppose not, that there is a vertex v with indeg(v) > outdeg(v).

There will be an edge left over! ⇒ not an euler tour.

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

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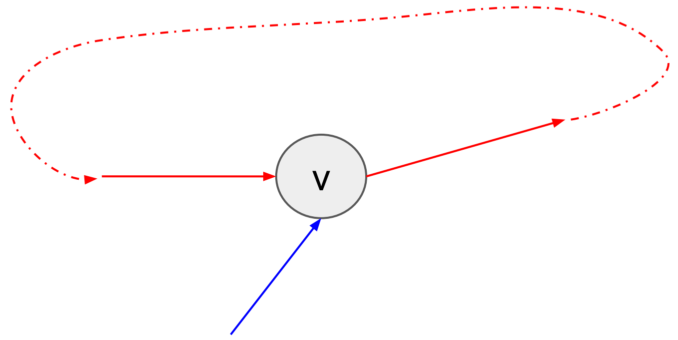
(→) Suppose G has an Euler tour.

We want to show every vertex v has indeg(v) = outdeg(v).



Suppose not, that there is a vertex v with indeg(v) > outdeg(v).

There will be an edge left over!

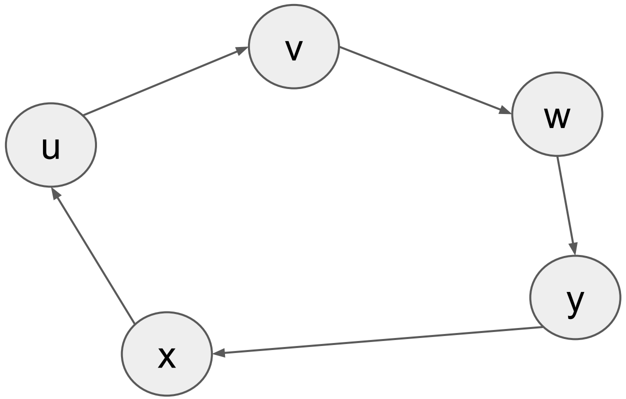
Exercise: show the same holds when indeg(v) < outdeg(v)

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

$$\text{in-degree}(v) = \text{out-degree}(v), \forall v \in V.$$

(←) Suppose indeg(v) = outdeg(v) for all vertices v.

We want to show there is an Euler tour



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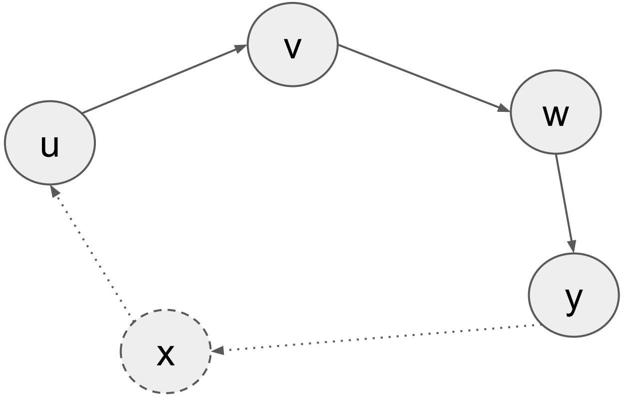

Suppose I delete a vertex (x)

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

$$\text{in-degree}(v) = \text{out-degree}(v), \forall v \in V.$$

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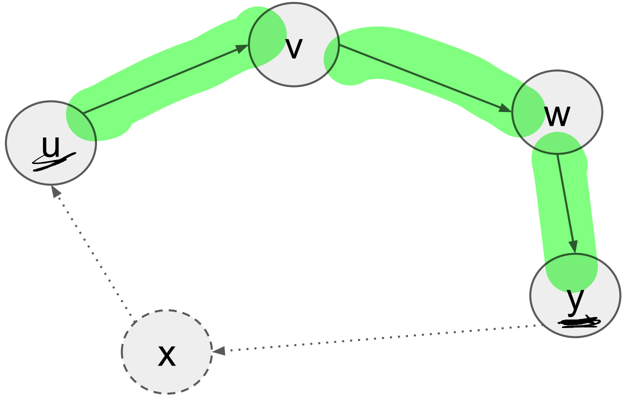
We want to show there is an Euler ~~tour~~ Path



Then there are vertices u,y such that:

indeg(y) = outdeg(y) + 1 = 1

indeg(u) = outdeg(u) - 1 = 0

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

$$\text{in-degree}(v) = \text{out-degree}(v), \forall v \in V.$$

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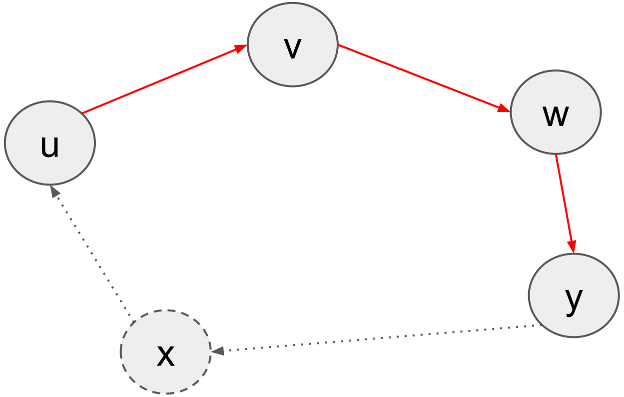

If we instead find an Euler **path** from $\underset{0}{u} \rightarrow \underset{1}{y}$,

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

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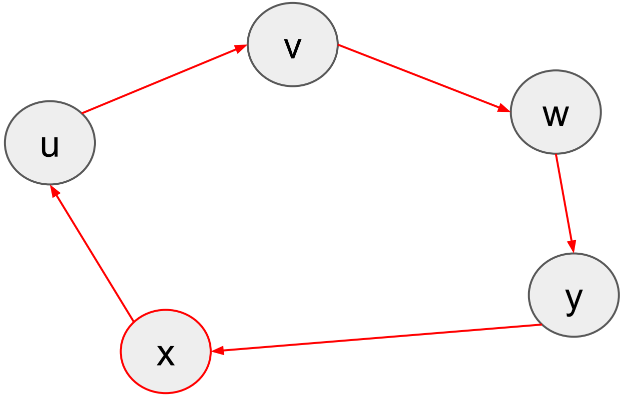

If we instead find an Euler **path** from u → y,

We can just add back x to get an Euler tour

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

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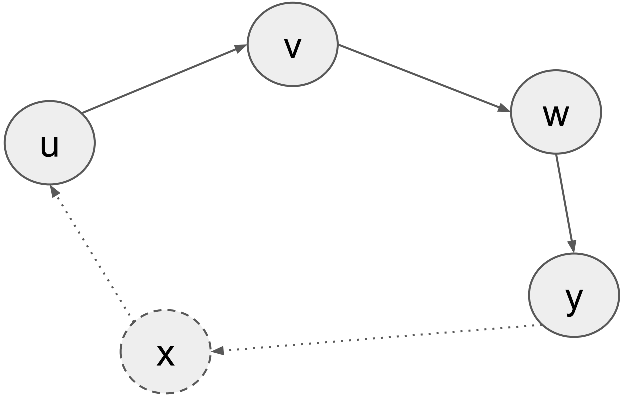

Then there are vertices u,y such that:

indeg(y) = outdeg(y) + 1

indeg(u) = outdeg(u) - 1

So let's find an Euler path in this graph

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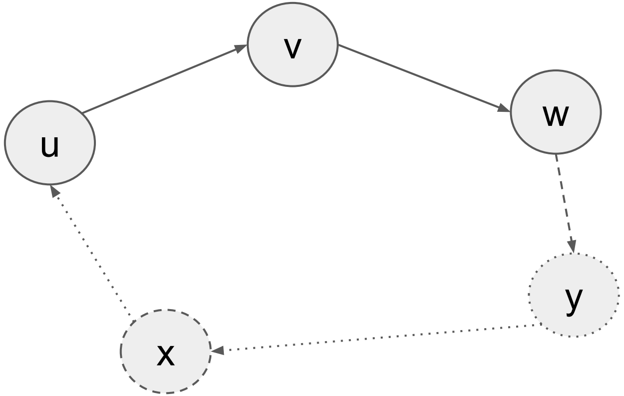

Then there are vertices u,y such that:

indeg(y) = outdeg(y) + 1

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Suppose I delete y

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

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We want to show there is an Euler tour

Then there are vertices u,y such that:

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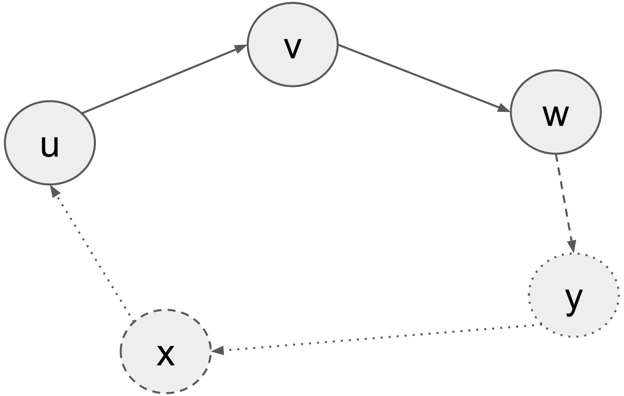

Then there are vertices u,w such that:

indeg(w) = outdeg(w) + 1

indeg(u) = outdeg(u) - 1

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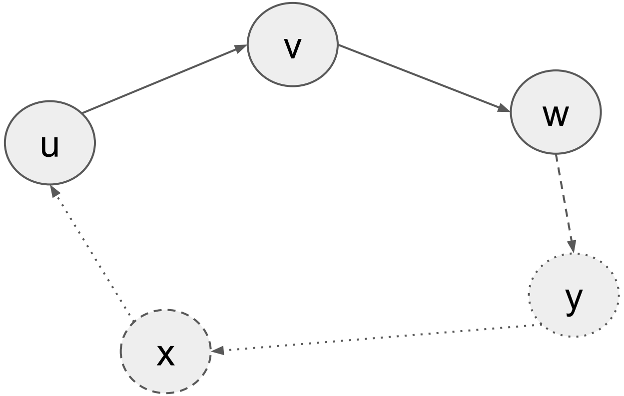

Then there are vertices u,y such that:

indeg(y) = outdeg(y) + 1

indeg(u) = outdeg(u) - 1

Then there are vertices u,w such that:

indeg(w) = outdeg(w) + 1

indeg(u) = outdeg(u) - 1

This new graph (deleted y) shares the same structure as the previous graph.. We can induct on the number of edges!

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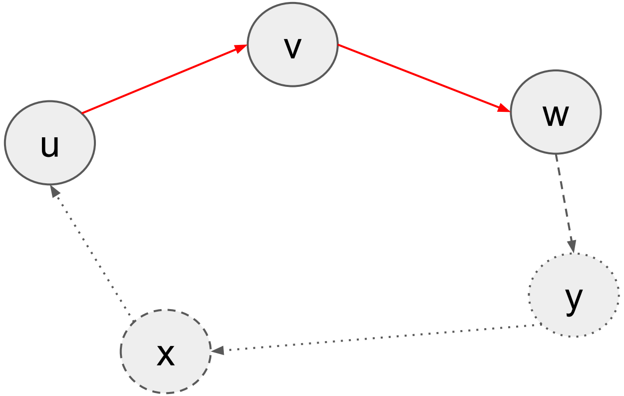
We want to show there is an Euler tour



Then there are vertices u,y such that:

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Then there are vertices u,w such that:

indeg(w) = outdeg(w) + 1

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By Induction there is an Euler path from u → w

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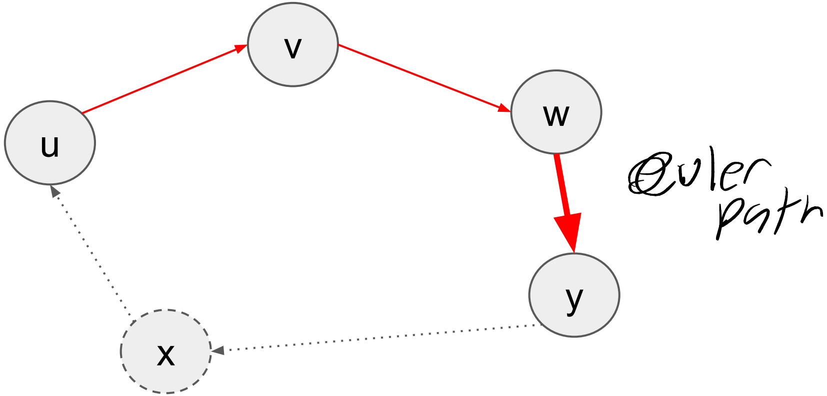
We want to show there is an Euler tour



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Then there are vertices u,w such that:

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Add back y

2. **(Euler tour)** An Euler tour of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has Euler tour if and only if

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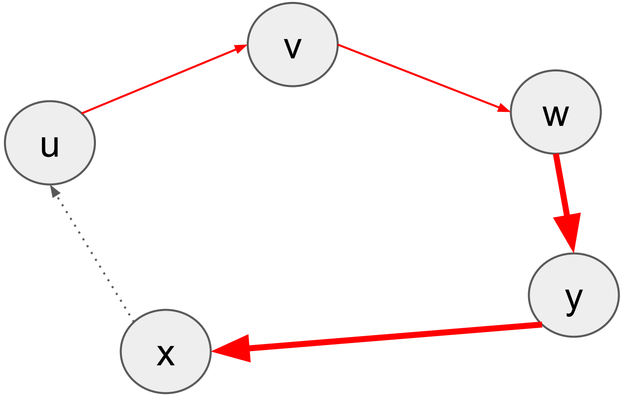

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Add back x

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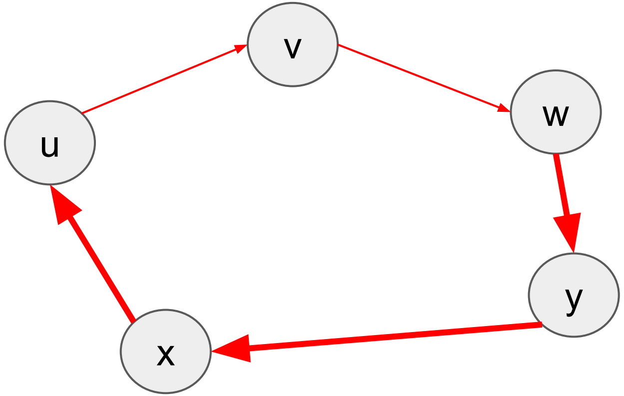
We want to show there is an Euler tour



Then there are vertices u,y such that:

indeg(y) = outdeg(y) + 1

indeg(u) = outdeg(u) - 1

Then there are vertices u,w such that:

indeg(w) = outdeg(w) + 1

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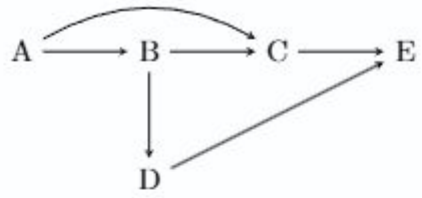
Complete the tour!

Question 2

Consider the directed graph $G = (V, E)$ given below:



1. Construct the adjacency matrix A of G.

u/v	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

Question 2

Consider the directed graph $G = (V, E)$ given below:

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left] & B \arrow[r] \arrow[d] & C \arrow[r] & E \\
 & D \arrow[urr] & &
\end{tikzcd}$$

2. Compute the transitive closure of G using Warshall's algorithm.

What's your algorithm Warshall/Floyd/Ingerman/Roy/Kleene?

History and naming [edit]

Worst-case space complexity	$\Theta(\lvert V\rvert^2)$

The Floyd–Warshall algorithm is an example of dynamic programming, and was published in its currently recognized form by Robert Floyd in 1962.[3] However, it is essentially the same as algorithms previously published by Bernard Roy in 1959[4] and also by Stephen Warshall in 1962[5] for finding the transitive closure of a graph,[6] and is closely related to Kleene's algorithm (published in 1956) for converting a deterministic finite automaton into a regular expression, with the difference being the use of a min-plus semiring.[7] The modern formulation of the algorithm as three nested for-loops was first described by Peter Ingerman, also in 1962.[8]

```
algorithm Floyd-Warshall(M:adjacency matrix representing G(V,E))

    R^(-1) ← M
    n ← |V|

    for k from 0 to n−1 do
        for i from 0 to n−1 do
            for j from 0 to n−1 do
                R^(k)[i,j] ← R^(k−1)[i,j] or (R^(k−1)[i,k] and R^(k−1)[k,j])
            end for
        end for
    end for

    return R^(n−1)
end algorithm
```

idea: iterate over all middle k

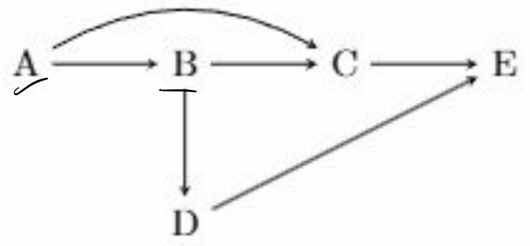
$i \to k \to j$

is there a path $i \to j$ going through k?

$R^k[i,j] = 1$ if we found a path between $i \to j$ on a prev. iter. ($R^{(k-1)}[i,j]$)

OR

There is a path $i \to k$ then $k \to j$ in a prev. iteration.

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    R^(-1) ← M
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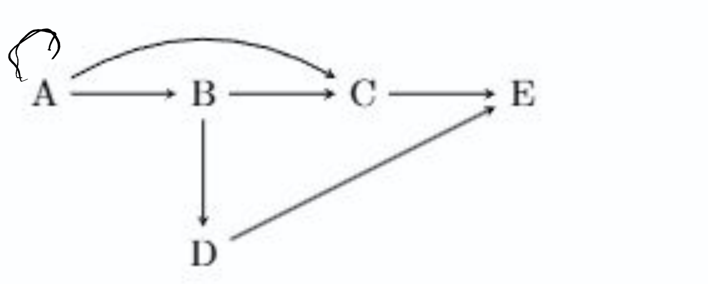
    for k from 0 to n − 1 do
       for i from 0 to n − 1 do
          for j from 0 to n − 1 do
             R^(k)[i, j] ← R^(k−1)[i, j] or (R^(k−1)[i, k] and R^(k−1)[k, j])
          end for
       end for
    end for

    return R^(n−1)
end algorithm
```

$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

$\xrightarrow{k = A}$

$R^{(0)}$	A	B	C	D	E
A					
B					
C					
D					
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
  for i **from** 0 **to** $n-1$ **do**
    for j **from** 0 **to** $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$
    end for
  end for
end for

return $R^{(n-1)}$
end algorithm

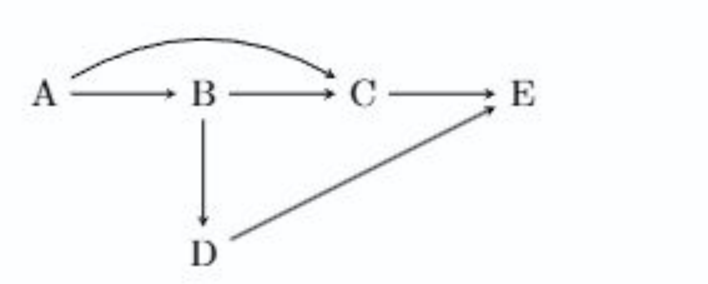
$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

is there a path A → A going through A.

$\xrightarrow{k = A}$

i = A
j = A

$R^{(0)}$	A	B	C	D	E
A					
B					
C					
D					
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
  for i **from** 0 **to** $n-1$ **do**
    for j **from** 0 **to** $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow \underbrace{R^{(k-1)}[i,j]}_{R^{(-1)}[a,a]=0}$ **or** $\underbrace{(R^{(k-1)}[i,k] \text{ and } R^{(k-1)}[k,j])}_{=0}$
    end for
  end for
end for

return $R^{(n-1)}$
end algorithm

$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

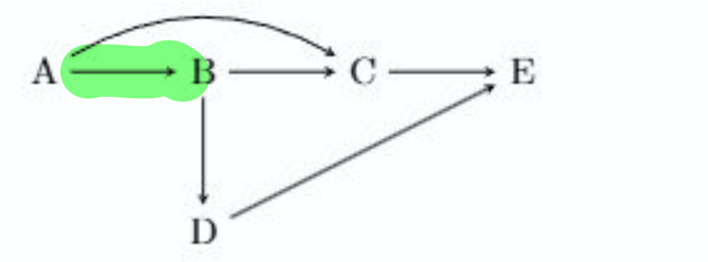
$\xrightarrow{k = A}$

i = A
j = A

$R^{(0)}$	A	B	C	D	E
A	0				
B					
C					
D					
E					

$A \to A \to B$

$\underline{\text{edge } (A, B)}$



$R^{(-1)} \leftarrow M$

$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
  for i **from** 0 **to** $n-1$ **do**
    for j **from** 0 **to** $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$
    end for
  end for
end for

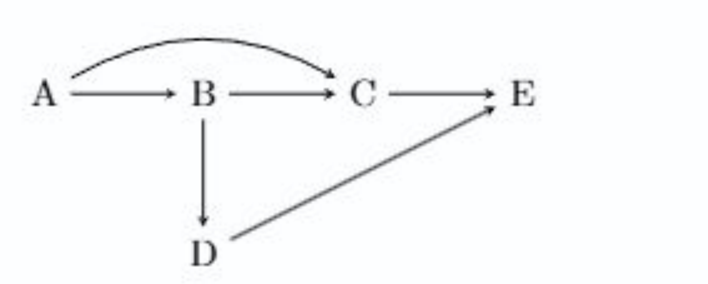
return $R^{(n-1)}$
end algorithm

$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

$\xrightarrow{k = A}$

i = A
j = B

$R^{(0)}$	A	B	C	D	E
A	0	1			
B					
C					
D					
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k from 0 to $n-1$ **do**
  for i from 0 to $n-1$ **do**
    for j from 0 to $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$
    end for
  end for
end for

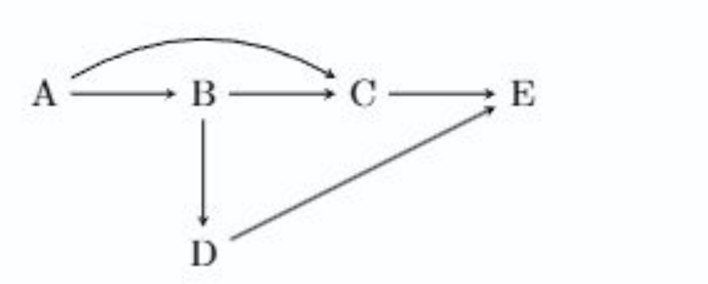
return $R^{(n-1)}$
end algorithm

$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

$\xrightarrow{k = A}$

i = A
j = C

$R^{(0)}$	A	B	C	D	E
A	0	1			
B					
C					
D					
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
  for i **from** 0 **to** $n-1$ **do**
    for j **from** 0 **to** $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$
    end for
  end for
end for

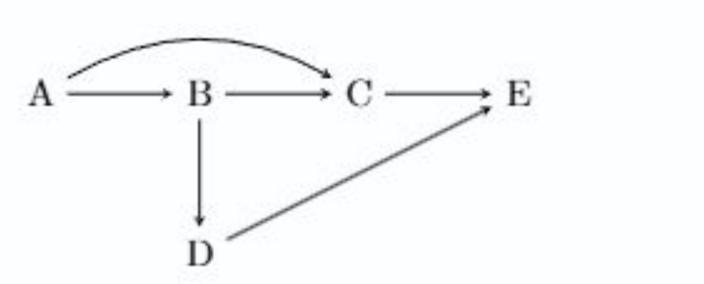
return $R^{(n-1)}$
end algorithm

$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

$\xrightarrow{k = A}$

i = A
j = C

$R^{(0)}$	A	B	C	D	E
A	0	1	1		
B					
C					
D					
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
  for i **from** 0 **to** $n-1$ **do**
    for j **from** 0 **to** $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$
    end for
  end for
end for

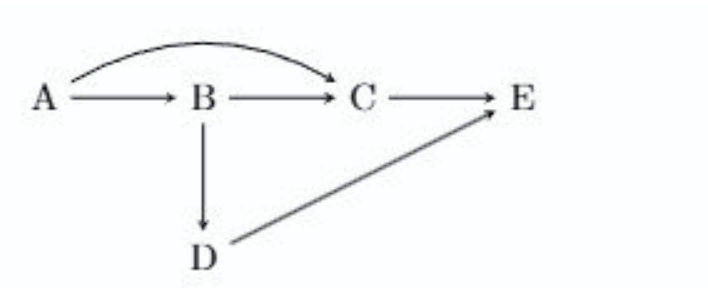
return $R^{(n-1)}$
end algorithm

$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

k = A

i = A
j = D

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	
B					
C					
D					
E					




```
    R^(−1) ← M
    n ← |V|

    for k from 0 to n−1 do
        for i from 0 to n−1 do
            for j from 0 to n−1 do
                R^(k)[i,j] ← R^(k−1)[i,j] or (R^(k−1)[i,k] and R^(k−1)[k,j])
            end for
        end for
    end for

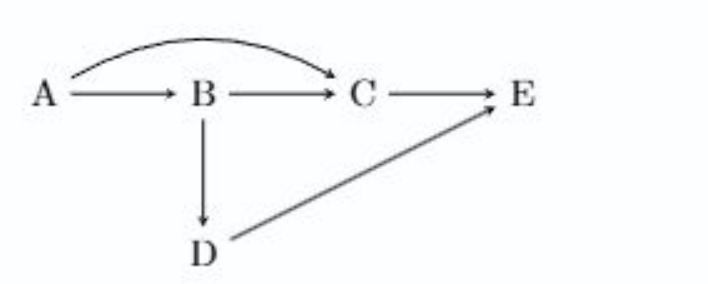
    return R^(n−1)
end algorithm
```

$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

$\xrightarrow[\substack{i = A \\ j = E}]{k = A}$

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B					
C					
D					
E					

no edge going into
A,




```
R^(-1) ← M
n ← |V|

for k from 0 to n−1 do
    for i from 0 to n−1 do
        for j from 0 to n−1 do
            R^(k)[i,j] ← R^(k−1)[i,j] or (R^(k−1)[i,k] and R^(k−1)[k,j])
        end for
    end for
end for

return R^(n−1)
end algorithm
```

$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

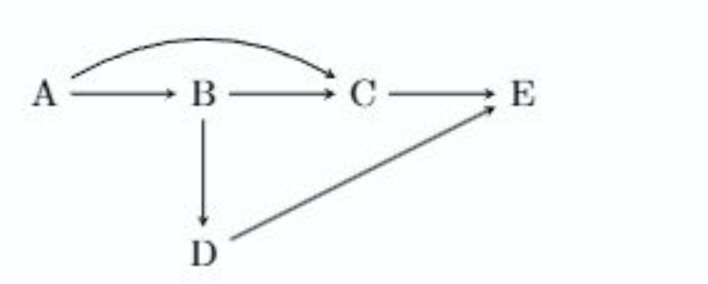
What's the final $R^{(0)}$?

Is there a path between B→A going through A

$\xrightarrow{k = A}$

i = B
j = A

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0				
C					
D					
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
  for i **from** 0 **to** $n-1$ **do**
    for j **from** 0 **to** $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$
    end for
  end for
end for

return $R^{(n-1)}$
end algorithm

$R^{(-1)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

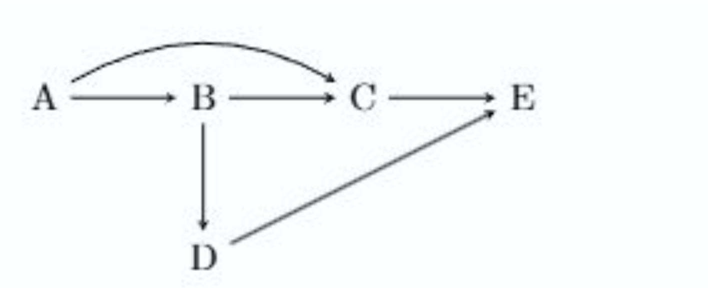
Whats the final $R^{(0)}$? Same as $R^{(-1)}$, why?

k = A

$\longrightarrow$

i = B
j = A

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k from 0 to $n-1$ **do**
  for i from 0 to $n-1$ **do**
    for j from 0 to $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$
    end for
  end for
end for

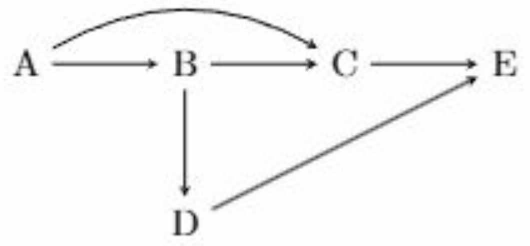
return $R^{(n-1)}$
end algorithm

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

$\xrightarrow{k = B}$

i = A
j = A

$R^{(1)}$	A	B	C	D	E
A	0	1	1		
B			1	1	
C					1
D					1
E					




```
R^(-1) ← M
n ← |V|

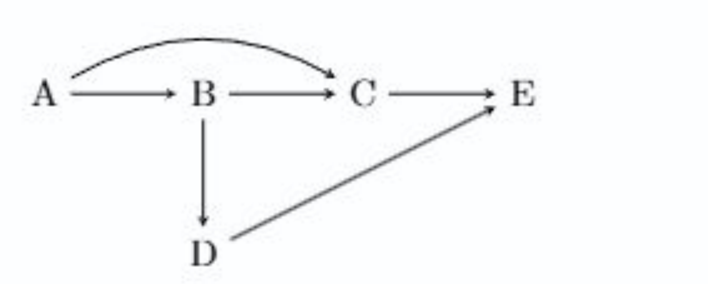
for k from 0 to n − 1 do
    for i from 0 to n − 1 do
        for j from 0 to n − 1 do
            R^(k)[i,j] ← R^(k−1)[i,j] or (R^(k−1)[i,k] and R^(k−1)[k,j])
        end for
    end for
end for

return R^(n−1)
end algorithm
```

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

$$\xrightarrow[\substack{i = A \\ j = B}]{k = B}$$

$R^{(1)}$	A	B	C	D	E
A	0	1	1		
B			1	1	
C					1
D					1
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k from 0 to $n-1$ **do**
  for i from 0 to $n-1$ **do**
    for j from 0 to $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$
    end for
  end for
end for

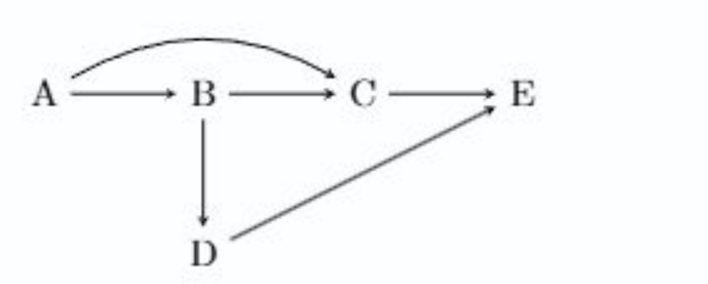
return $R^{(n-1)}$
end algorithm

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

k = B

i = A
j = C

$R^{(1)}$	A	B	C	D	E
A	0	1	1		
B			1	1	
C					1
D					1
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k from 0 to $n-1$ **do**
  for i from 0 to $n-1$ **do**
    for j from 0 to $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$
    end for
  end for
end for

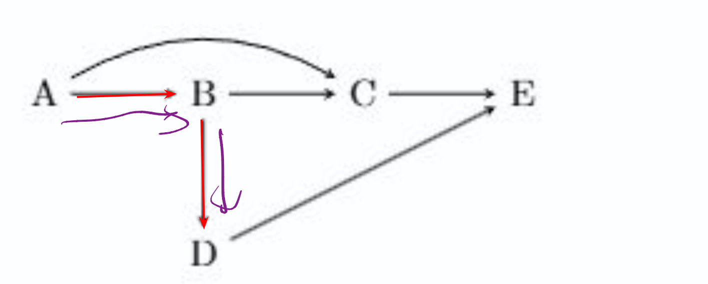
return $R^{(n-1)}$
end algorithm

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

k = B

i = A
j = D

$R^{(1)}$	A	B	C	D	E
A	0	1	1		
B			1	1	
C					1
D					1
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
  for i **from** 0 **to** $n-1$ **do**
    for j **from** 0 **to** $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$
    end for
  end for
end for

return $R^{(n-1)}$
end algorithm

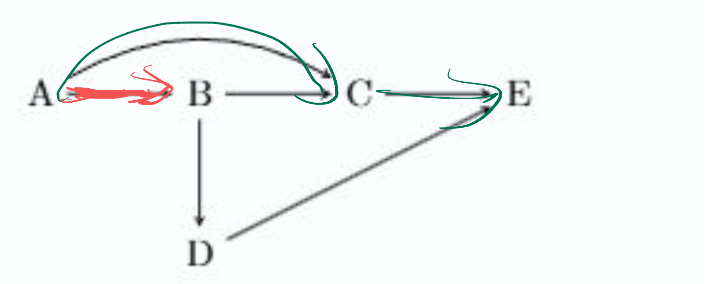
V
0

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

k = B

i = A
j = D

$R^{(1)}$	A	B	C	D	E
A	0	1	1	1	
B			1	1	
C					1
D					1
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
    for i **from** 0 **to** $n-1$ **do**
        for j **from** 0 **to** $n-1$ **do**
            $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$
        end for
    end for
end for

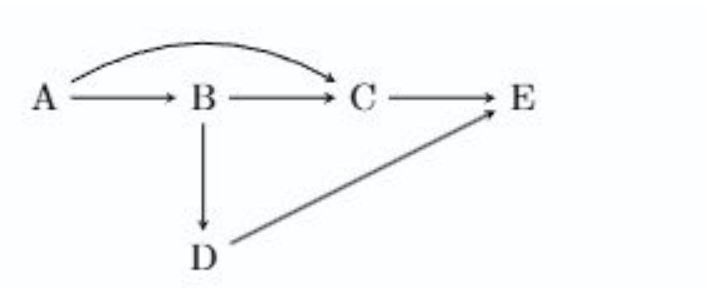
return $R^{(n-1)}$
end algorithm

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

$k = B$ $\longrightarrow$

$i = A$
$j = E$

$R^{(1)}$	A	B	C	D	E
A	0	1	1	1	? → 0
B			1	1	
C					1
D					1
E					




```
R^(-1) ← M
n ← |V|

for k from 0 to n−1 do
    for i from 0 to n−1 do
        for j from 0 to n−1 do
            R^(k)[i,j] ← R^(k−1)[i,j] or (R^(k−1)[i,k] and R^(k−1)[k,j])
        end for
    end for
end for

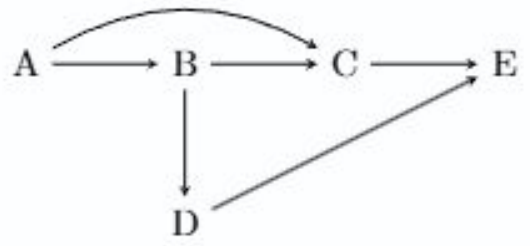
return R^(n−1)
end algorithm
```

$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

$$\xrightarrow{k = B}$$

i = A
j = E

$R^{(1)}$	A	B	C	D	E
A	0	1	1	1	0
B			1	1	
C					1
D					1
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k from 0 to $n-1$ **do**
  for i from 0 to $n-1$ **do**
    for j from 0 to $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$
    end for
  end for
end for

return $R^{(n-1)}$
end algorithm

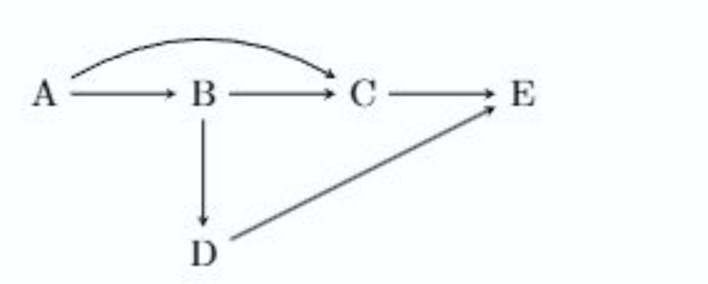
$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

What's the final $R^{(1)}$?

$\xrightarrow{k = B}$

i = B
j = A

$R^{(1)}$	A	B	C	D	E
A	0	1	1	1	0
B			1	1	
C					1
D					1
E					




```
R^(-1) ← M
n ← |V|

for k from 0 to n−1 do
   for i from 0 to n−1 do
      for j from 0 to n−1 do
         R^(k)[i,j] ← R^(k−1)[i,j] or (R^(k−1)[i,k] and R^(k−1)[k,j])
      end for
   end for
end for

return R^(n−1)
end algorithm
```

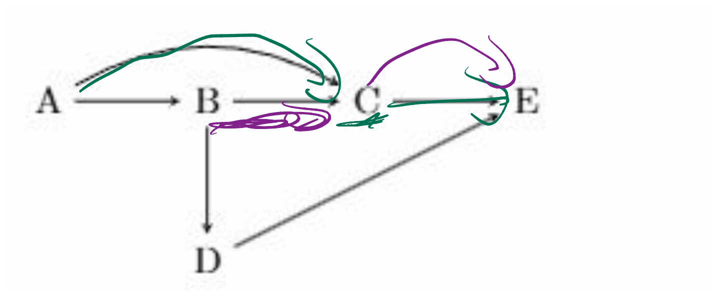
$R^{(0)}$	A	B	C	D	E
A	0	1	1	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

What's the final $R^{(1)}$?

k = B

i = B
j = A

$R^{(1)}$	A	B	C	D	E
A	0	1	1	1	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0



$R^{(-1)} \leftarrow M$

$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
  for i **from** 0 **to** $n-1$ **do**
    for j **from** 0 **to** $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$
    end for
  end for
end for

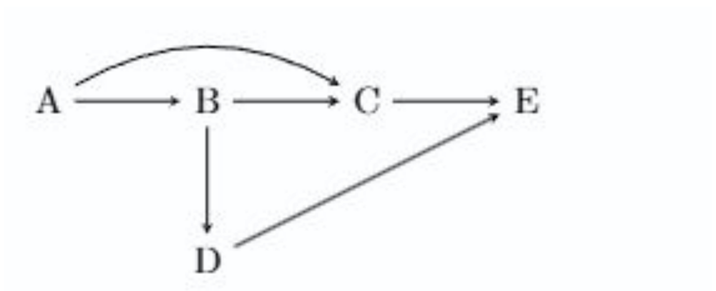
return $R^{(n-1)}$
end algorithm

$R^{(1)}$	A	B	C	D	E
A	0	1	1	1	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

What's the final $R^{(2)}$?

k = C

$R^{(2)}$	A	B	C	D	E
A					1
B					
C					
D					
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k **from** 0 **to** $n-1$ **do**
  for i **from** 0 **to** $n-1$ **do**
    for j **from** 0 **to** $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$
    end for
  end for
end for

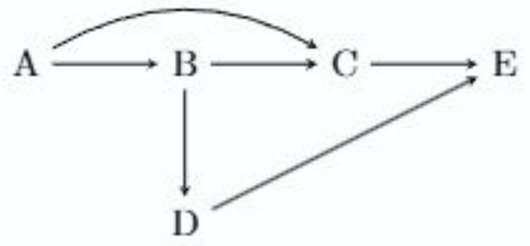
return $R^{(n-1)}$
end algorithm

$R^{(1)}$	A	B	C	D	E
A	0	1	1	1	0
B	0	0	1	1	0
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

What's the final $R^{(2)}$?

$k = C$

$R^{(2)}$	A	B	C	D	E
A	0	1	1	1	1
B	0	0	1	1	1
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0




```
R^(-1) ← M
n ← |V|

for k from 0 to n−1 do
   for i from 0 to n−1 do
      for j from 0 to n−1 do
         R^(k)[i,j] ← R^(k−1)[i,j] or (R^(k−1)[i,k] and R^(k−1)[k,j])
      end for
   end for
end for

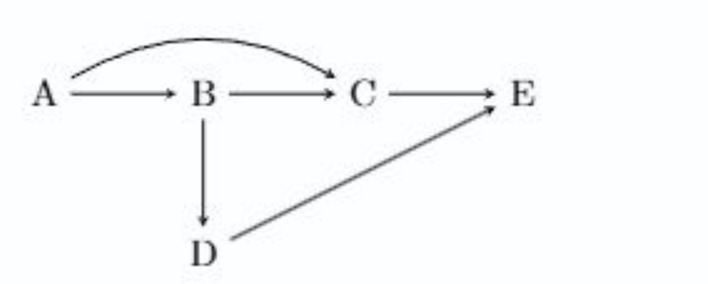
return R^(n−1)
end algorithm
```

$R^{(2)}$	A	B	C	D	E
A	0	1	1	1	1
B	0	0	1	1	1
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

What's the final $R^{(3)}$?

k = D

$R^{(3)}$	A	B	C	D	E
A					
B					
C					
D					
E					



$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k from 0 to $n-1$ **do**
  for i from 0 to $n-1$ **do**
    for j from 0 to $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$
    end for
  end for
end for

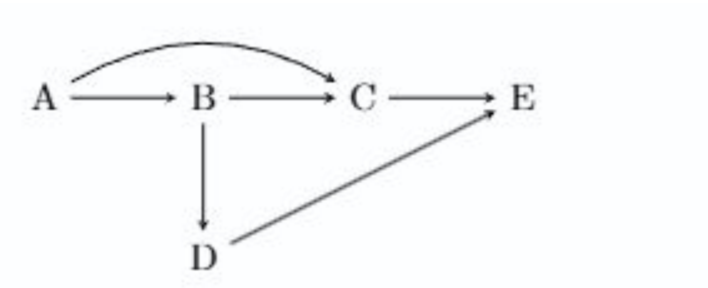
return $R^{(n-1)}$
end algorithm

$R^{(2)}$	A	B	C	D	E
A	0	1	1	1	1
B	0	0	1	1	1
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

What's the final $R^{(3)}$?

k = D

$R^{(3)}$	A	B	C	D	E
A	0	1	1	1	1
B	0	0	1	1	1
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0



$R^{(-1)} \leftarrow M$

$n \leftarrow |V|$

```
for k from 0 to n−1 do
   for i from 0 to n−1 do
      for j from 0 to n−1 do
         R(k)[i,j] ← R(k−1)[i,j] or (R(k−1)[i,k] and R(k−1)[k,j])
      end for
   end for
end for

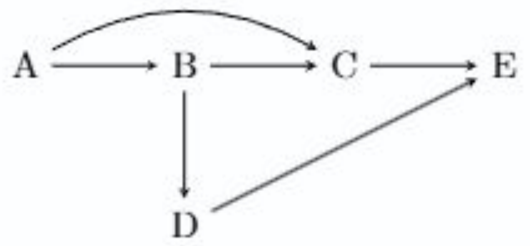
return R(n−1)
end algorithm
```

$R^{(3)}$	A	B	C	D	E
A	0	1	1	1	1
B	0	0	1	1	1
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

What's the final $R^{(4)}$?

$\xrightarrow{k = E}$

$R^{(4)}$	A	B	C	D	E
A	0	1	1	1	1
B	0	0	1	1	1
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

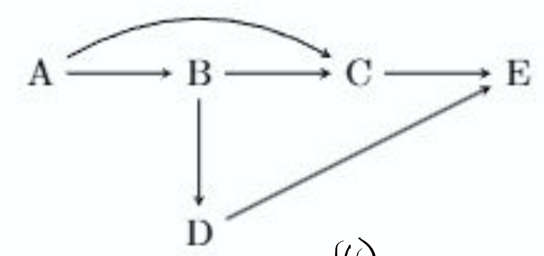


$R^{(4)}$	A	B	C	D	E
A	0	1	1	1	1
B	0	0	1	1	1
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

Summary

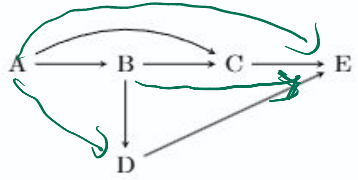
For each k = A,B,C,D,E:
    For each i = ….:
        For each j = …:
            Check if there is a path between i and j through k

$R^{(-1)}$: Adj Matrix
$R^{(0/A)}$: $R^{(-1)}$ + (paths through A)
$R^{(1/B)}$: $R^{(0/A)}$+ (paths through B)
$R^{(2/C)}$: $R^{(1/B)}$ + (paths through C)
$R^{(3/D)}$: $R^{(2/C)}$+ (paths through D)
$R^{(4/E)}$: $R^{(3/D)}$+ (paths through E)



Question 2

Consider the directed graph $G = (V, E)$ given below:



3. Draw the graph representation of the transitive closure of G.

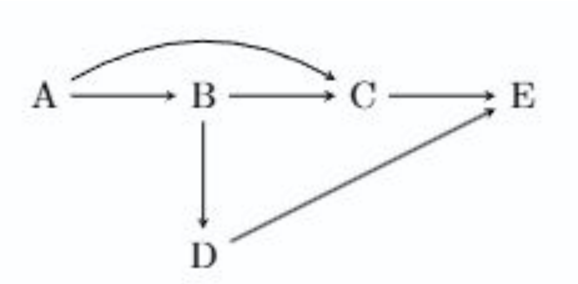
$R^{(4)}[i,j] = 1$ iff there is a path $i \rightsquigarrow j$.

$R^{(4)}$	A	B	C	D	E
A	0	1	1	1	1
B	0	0	1	1	1
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0

Transitive Closure(G):

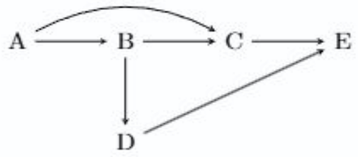
add edge (i,j) to G

if there is a path $i \rightsquigarrow j$



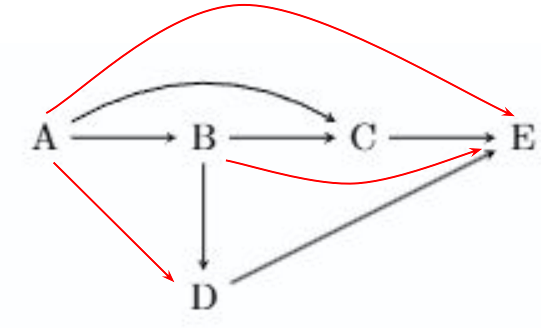
Question 2

Consider the directed graph $G = (V, E)$ given below:



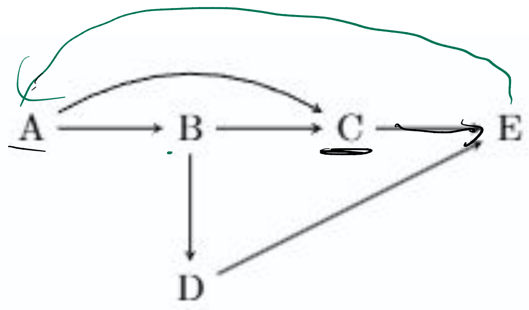
4. Determine the reachability of each node in G.

$R^{(4)}$	A	B	C	D	E
A	0	1	1	1	1
B	0	0	1	1	1
C	0	0	0	0	1
D	0	0	0	0	1
E	0	0	0	0	0



The transitive closure can help here

Check if edge exists
in transitive closure.



5. Identify if G is strongly connected. If not, can you add one edge to make G become a strongly connected graph?

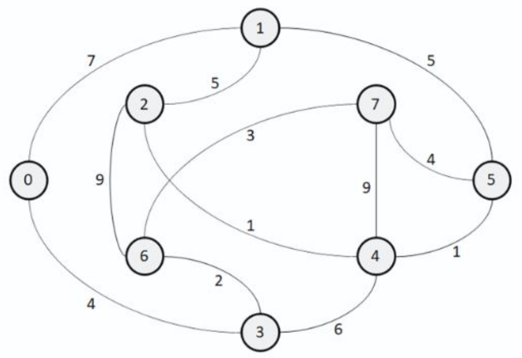
no, no path between $E \to A$.

- A is a source (indeg(A) = 0 but it goes to all other vertices)
- E is a sink (outdeg(E) = 0 but all other vertices go to it)

connect sink $\to$ source to make G strongly connected.

Question 3

Consider the following graph G:



Let G_d be a directed graph using the vertices of G. For a pair of vertices u and v connected by an edge in G, their respective directed edge in G_d is as follows:

$$\text{Edge with vertices } u \text{ and } v = \begin{cases} (u, v), & \deg(u) < \deg(v) \lor (\deg(u) = \deg(v) \land u < v) \\ (v, u), & \text{Otherwise} \end{cases}$$

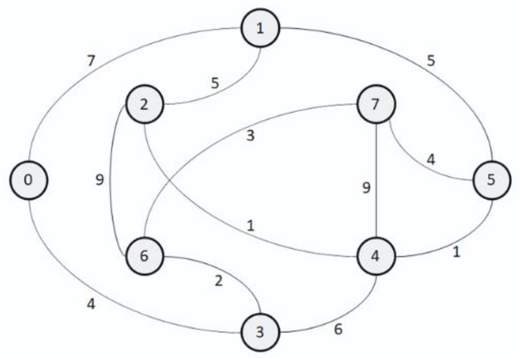
Let's draw G_d

First, calculate deg(v)

v	deg(v)
1	
2	
3	
4	
5	
6	
7	

Question 3

Consider the following graph G:



Let G_d be a directed graph using the vertices of G. For a pair of vertices u and v connected by an edge in G, their respective directed edge in G_d is as follows:

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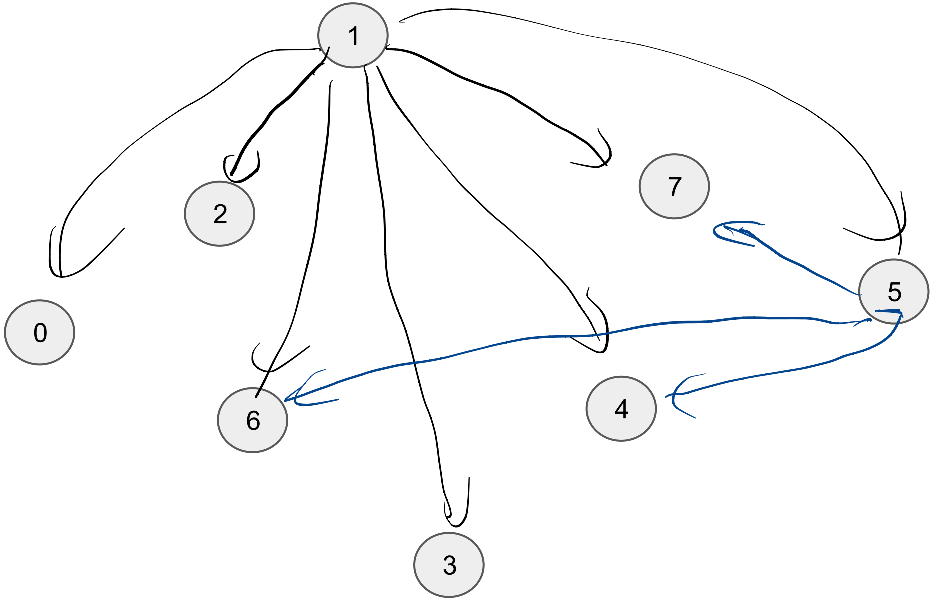
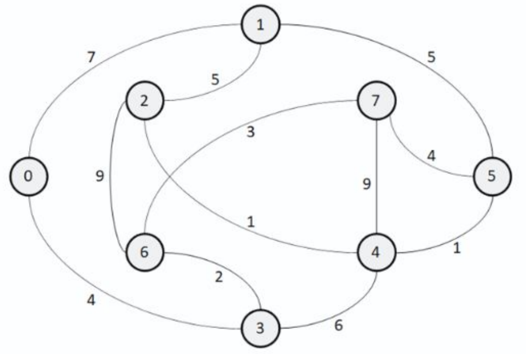
v	deg(v)
1	3
2	3
3	3
4	4
5	3
6	3
7	3

Let's draw G_d



Question 3

Consider the following graph G:



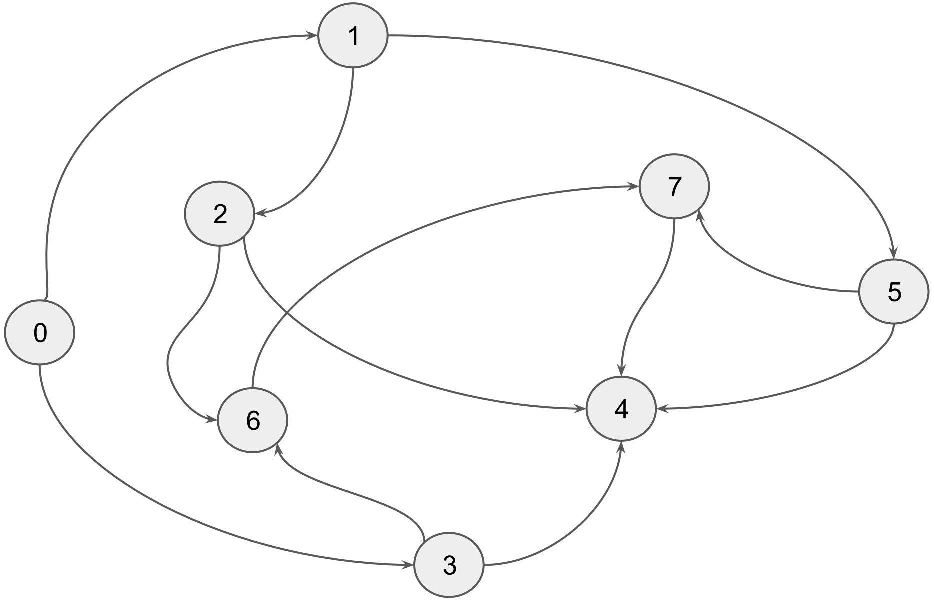
Let G_d be a directed graph using the vertices of G. For a pair of vertices u and v connected by an edge in G, their respective directed edge in G_d is as follows:

$$\text{Edge with vertices } u \text{ and } v = \begin{cases} (u, v), & \deg(u) < \deg(v) \vee (\deg(u) = \deg(v) \wedge u < v) \\ (v, u), & \text{Otherwise} \end{cases}$$

Let's draw G_d

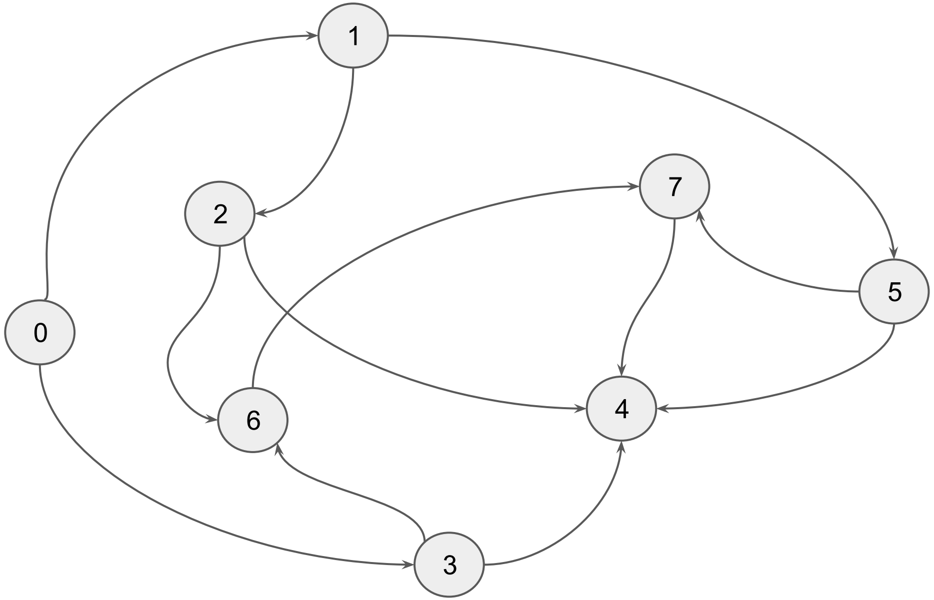
deg (v) =

v	deg(v)
1	3
2	3
3	3
4	4
5	3
6	3
7	3



1. Is G_d strongly connected? If yes, explain why. Otherwise, list the minimum number of edges required to make G_d strongly connected.

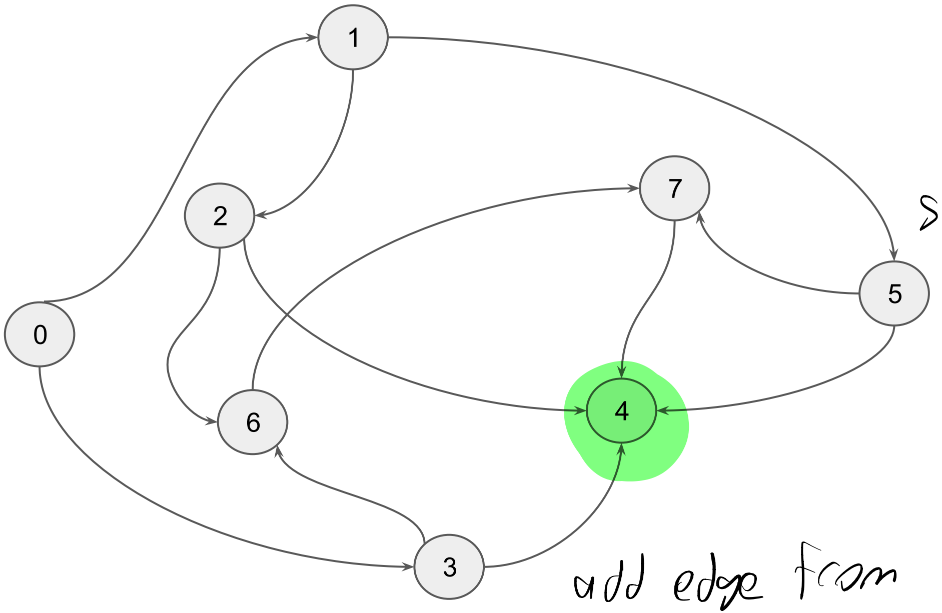
v	deg(v)
1	3
2	3
3	3
4	4
5	3
6	3
7	3



1. Is G_d strongly connected? If yes, explain why. Otherwise, list the minimum number of edges required to make G_d strongly connected.

no

If we run Warshall..



sink

source: looked at row

sink by looking at col.

u/v	0	1	2	3	4	5	6	7
0		1	1	1	1	1	1	1
1			1		1	1	1	1
2					1		1	1
3					1		1	1
4	0	0	0	0	0	0	0	0
5					1			1
6					1			1
7	0	0	0	0	1	0	0	0

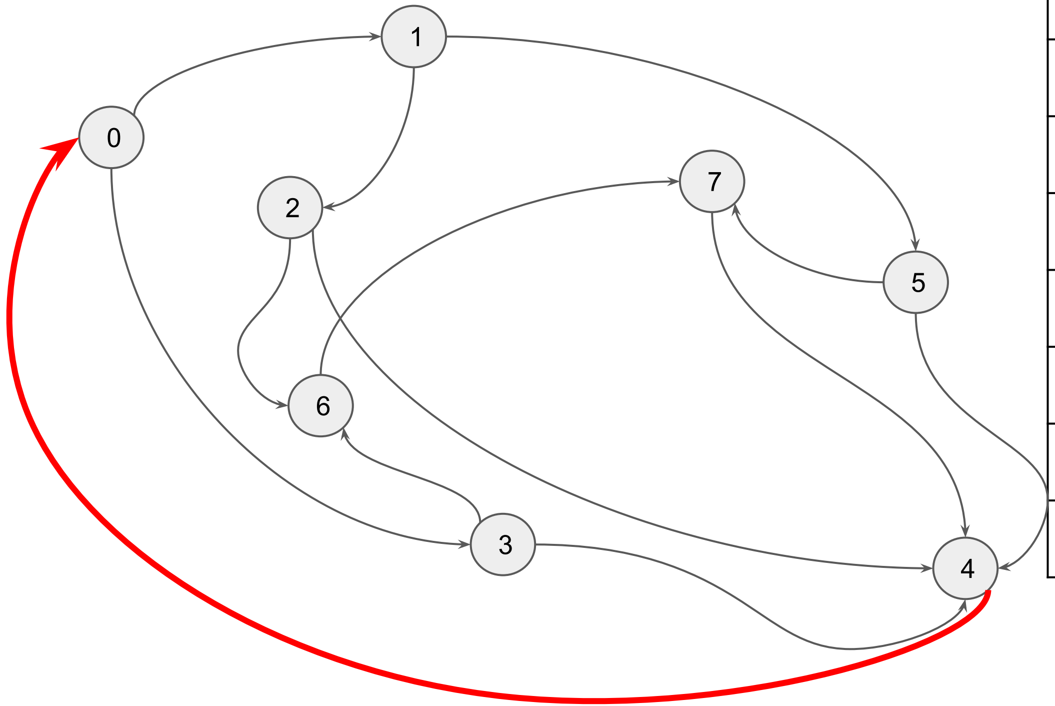
add edge from $4 \to 0$

There is no path $4 \to 0$

Observations:

1. Is G_d strongly connected? If yes, explain why. Otherwise, list the minimum number of edges required to make G_d strongly connected.

If we run Warshall..

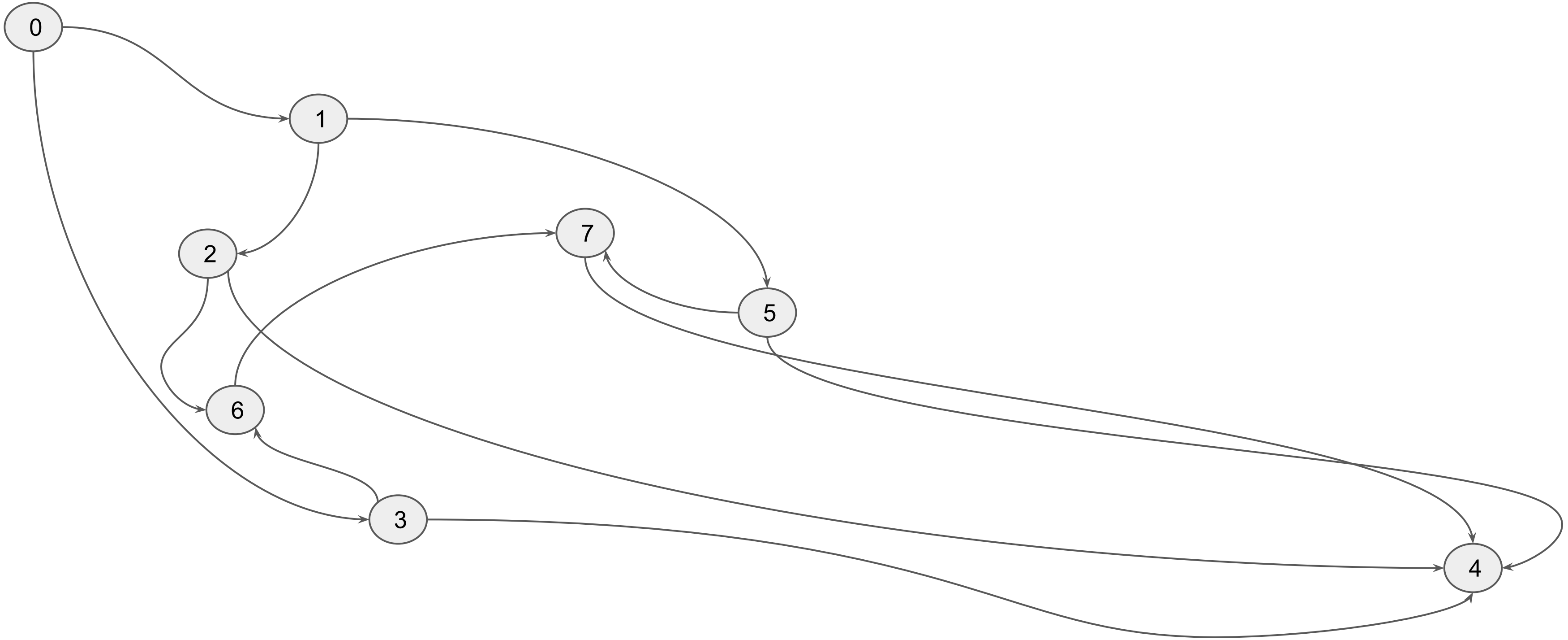


u/v	0	1	2	3	4	5	6	7
0		1	1	1	1	1	1	1
1			1		1	1	1	1
2					1		1	1
3					1		1	1
4								
5					1			1
6					1			1
7					1			

Adding (4,0) makes strongly connected

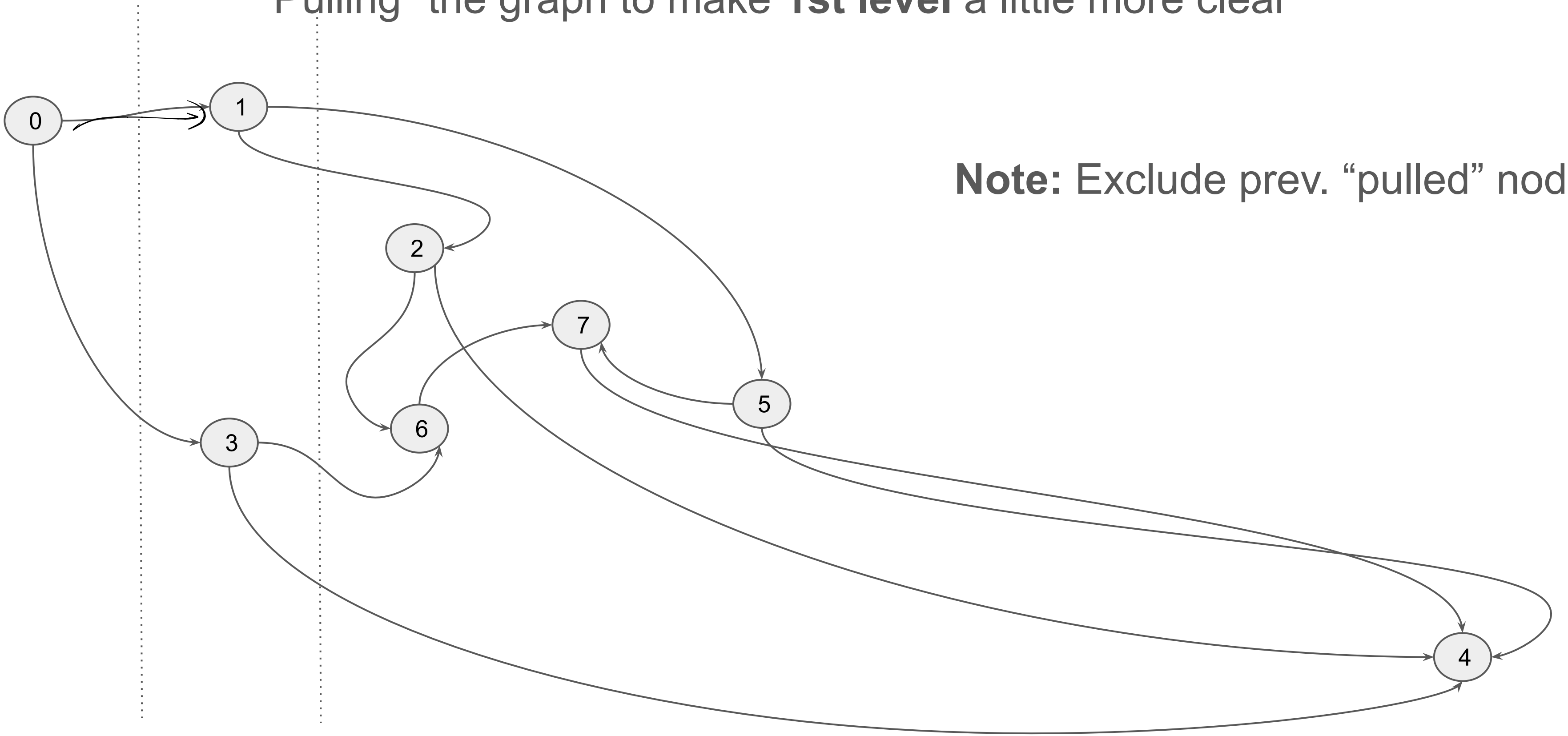
2. Show all the topological orderings of G_d.

“Pulling” the graph to make source/sink a little more clear



2. Show all the topological orderings of G_d.

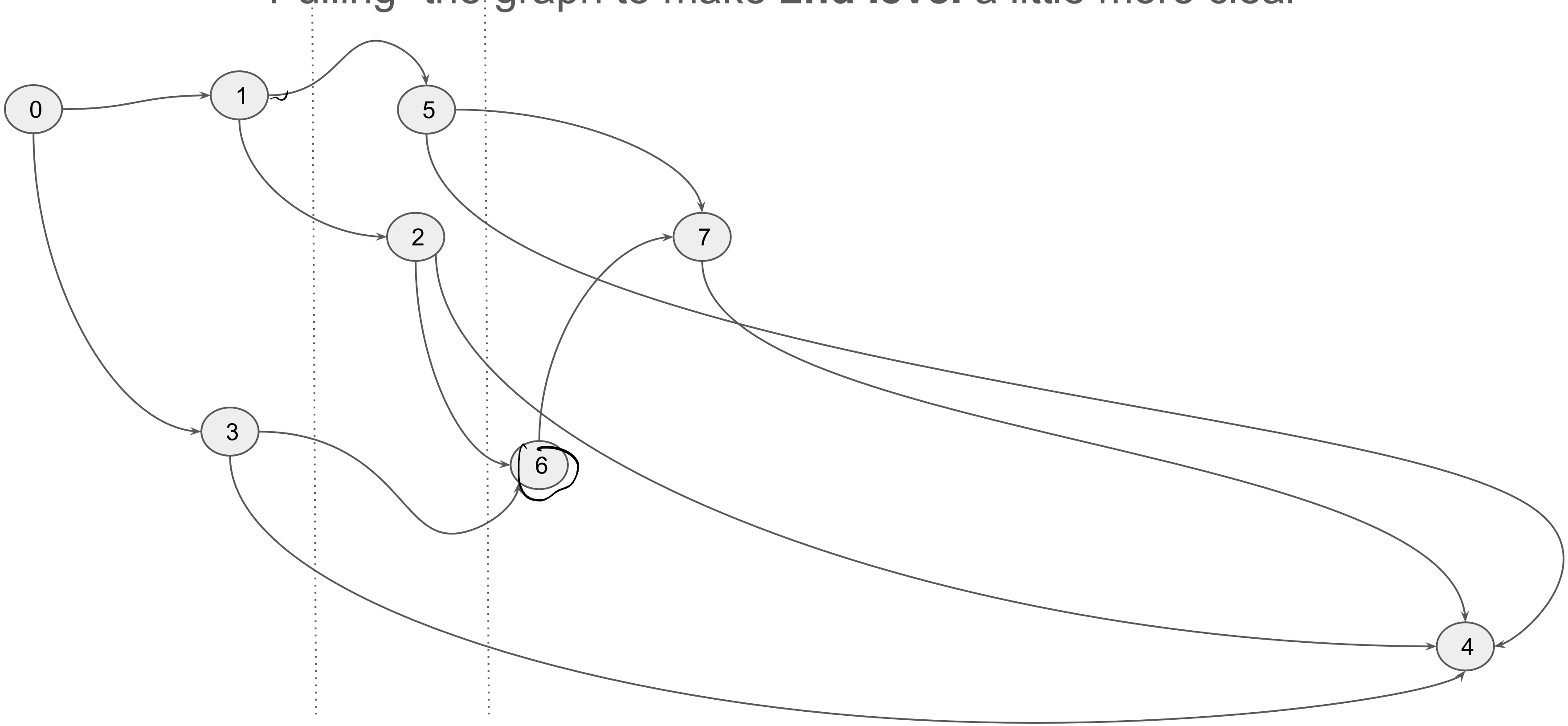
"Pulling" the graph to make **1st level** a little more clear



Note: Exclude prev. "pulled" nodes

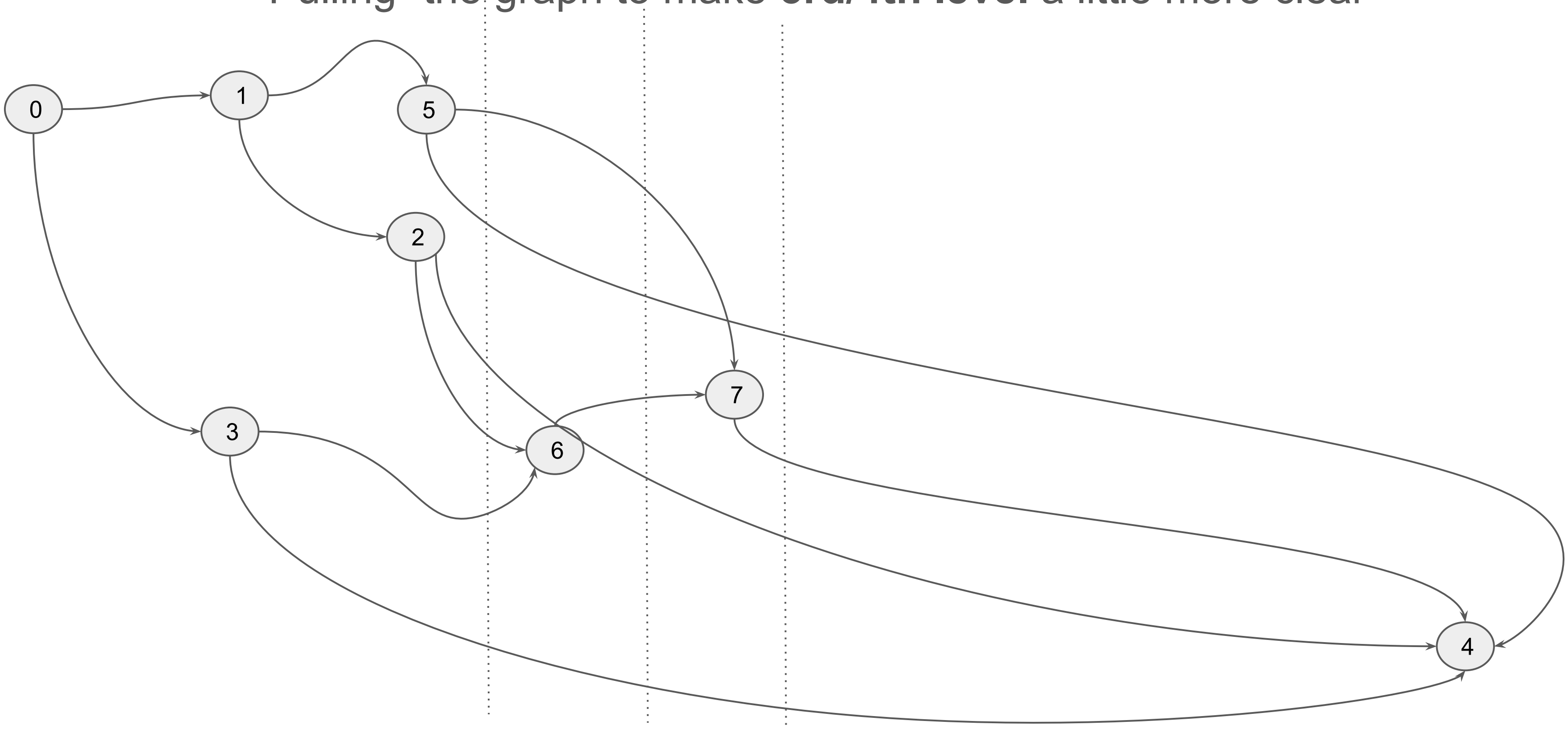
2. Show all the topological orderings of G_d.

"Pulling" the graph to make **2nd level** a little more clear



2. Show all the topological orderings of G_d.

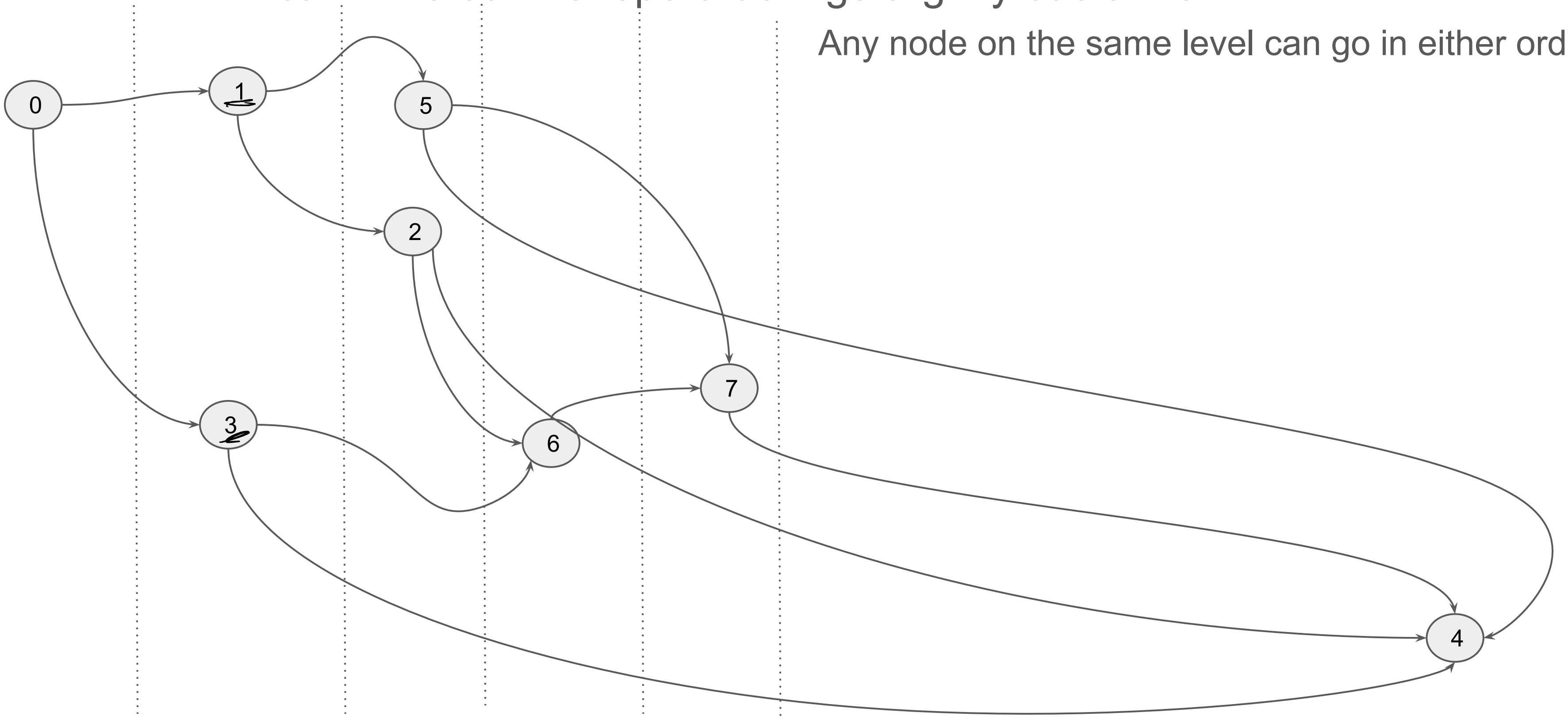
"Pulling" the graph to make **3rd/4th level** a little more clear



2. Show all the topological orderings of G_d.

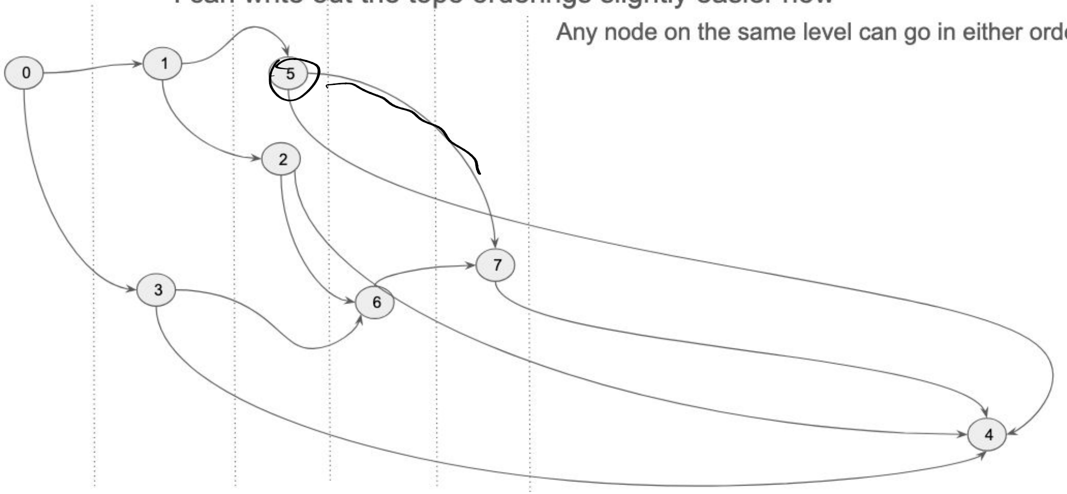
I can write out the topo orderings slightly easier now

Any node on the same level can go in either order



I can write out the topo orderings slightly easier now

Any node on the same level can go in either order



Idea: chain by chain

0 → 1 → 5 → 2 → 6 → 7 → 4

3 → 5 → 2 → 6 → 7 → 4

3 → 1 → 5 → 2 → 6 → 7 → 4